## Warm-up

**Problem 1.** Try various parameters for Theorem 60: plot, for $\beta \in (0,1)$, the bound, when

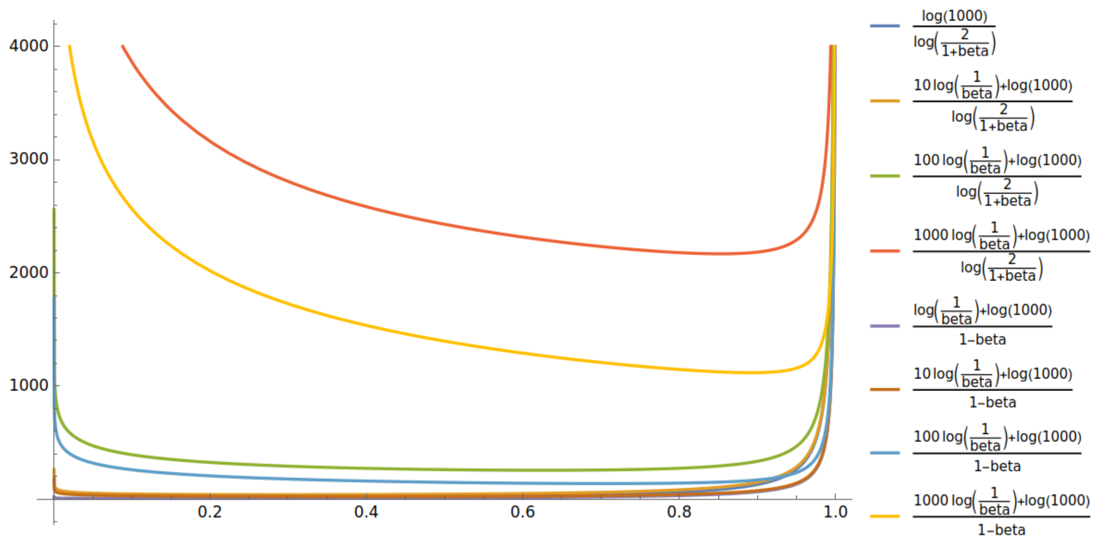
- $C^* = 0, n = 1000$
- $C^* = 10, n = 1000$
- $C^* = 100, n = 1000$
- $C^* = 1000, n = 1000$

Do the same for Theorem 61.

**Solution 1.** Try it at home! Here in Mathematica:

```
In[3]:= Plot[{ (Log[1000]) / Log[2 / (1 + beta)],
     (10 * Log[1 / beta] + Log[1000]) / Log[2 / (1 + beta)],
     (100 * Log[1 / beta] + Log[1000]) / Log[2 / (1 + beta)],
     (1000 * Log[1 / beta] + Log[1000]) / Log[2 / (1 + beta)],
     (Log[1 / beta] + Log[1000]) / (1 - beta),
     (10 * Log[1 / beta] + Log[1000]) / (1 - beta),
     (100 * Log[1 / beta] + Log[1000]) / (1 - beta),
     (1000 * Log[1 / beta] + Log[1000]) / (1 - beta)},
    {beta, 0, 1}, PlotLegends → "Expressions"]
```

Out[3]=




**Problem 2.** Assume you know both $n$ and (an upper bound on) $C^*$ in advance. How would you set $\beta$ in the MWU? In the Randomised MWU?

**Solution 2.** Given explicit values: differentiate the expressions to find the minimum (or find the minimum numerically).

To find a reasonable approximation (up to a factor 2): choose $\beta$ to balance the two terms in the numerator,

$$C^* \log(1/\beta) = \log n$$

This might not be th exact minimum, but will be within a constant factor, and is much simpler to derive.

**Problem 3.** Prove Fact 56.3: namely, consider $n = 2$ experts, one predicting always 0 and the other always 1, and consider all $2^T$ possible sequences $(u_1, \ldots, u_T)$. Show that for any deterministic algorithm $A$, there exists a sequence on which the algorithm makes $T$ mistakes, and use this to conclude.

**Solution 3.** Same reasoning and construction as Fact 56.1, but need to also show that $C^*(T)$ is at most $T/2$ (best expert will make at most $T/2$ mistakes). Note that since one expert always outputs 0 and the other always 1, the best expert will make at most $T/2$ mistakes: denote $S_1 = \{t \in [T] : u_t = 0\}$ and $S_2 = [T] \backslash S_1$, expert 1 will be correct $|S_1|$ times and expert 2 correct for $|S_2|$ times –

$$\max\{|S_1|, |S_2|\} = \max\{|S_1|, T - |S_1|\} \geqslant T/2$$

Suppose given algorithm $A$ (you can predict what it will output every step)

1) observes $\{0, 1\}$, predict $A$'s output $\hat{u}_1$. Set $u_1 \leftarrow 1 - \hat{u}_1$.
2) observes $\{u_1, 0, 1\}$, predict $A$'s output $\hat{u}_2$. Set $u_2 \leftarrow 1 - \hat{u}_2$.
3) observes $\{u_1, u_2, 0, 1\}$, predict $A$'s output $\hat{u}_3$. Set $u_3 \leftarrow 1 - \hat{u}_3$.
etc. until $T$.

So your algorithm will make $T$ mistakes while best expert will make at most $T/2$. The factor 2 is necessary for deterministic algorithms.

## Problem solving

**Problem 4.** Suppose $C^* = C^*(T)$ is known in advance. We will show how to modify the MWU algorithm to achieve

$$C(T) \leq 2C^* + O(\sqrt{C^*(T) \log n} + \log n)$$

a) Argue that, if $C^* \leq \log n$, we are done.
b) Suppose $C^* > \log n$. Show how to achieve the desired bound by setting $\beta = 1 - \varepsilon$, for some suitable $\varepsilon = \varepsilon(C^*, n)$.
c) Conclude.

**Solution 4.**

a) If $C^* \leqslant 4\log n$, then we set $\beta = \frac{1}{2}$.

$$\frac{C^* \log \frac{1}{\beta} + \log n}{\log \frac{2}{1+\beta}} \leqslant 2.41(C^* + \log n) \leqslant O(\log n).$$

b) We need some approximation facts first. For $\varepsilon \in (0, 0.5)$, we have $\varepsilon \leqslant \log \frac{1}{1-\varepsilon} \leqslant 2\varepsilon$ and

$$\frac{\log \frac{1}{1-\varepsilon}}{\log \frac{1}{1-\varepsilon/2}} = 2 + \frac{\varepsilon}{2} + O(\varepsilon^2) \leqslant 2 + \varepsilon.$$

by series expansion. We restrict $1 - \beta = \varepsilon < \frac{1}{2}$. We proceed with the calculation:

$$\begin{aligned}
C^* \frac{\log \frac{1}{\beta}}{\log \frac{2}{1+\beta}} + \frac{\log n}{\log \frac{2}{1+\beta}} &= C^* \frac{\log \frac{1}{1-\varepsilon}}{\log \frac{1}{1-\varepsilon/2}} + \frac{\log n}{\log \frac{1}{1-\varepsilon/2}} \\
&\leqslant C^* \cdot (2+\varepsilon) + \frac{\log n}{\varepsilon} \\
&= 2C^* + C^*\varepsilon + \frac{\log n}{\varepsilon} \\
&\leqslant 2C^* + C^*\sqrt{\log n / C^*} + \frac{\log n}{\sqrt{\log n / C^*}} \\
&= 2C^* + 2\sqrt{C^* \log n}
\end{aligned}$$

Note that we need $\varepsilon = \sqrt{\log n / C^*} < \frac{1}{2}$, or, equivalently, $\frac{\log n}{C^*} < \frac{1}{4}$. (This motivates in hindsight question a)).

c) Combining the two, we have that

$$\frac{C^* \log \frac{1}{\beta} + \log n}{\log \frac{2}{1+\beta}} \leqslant 2C^* + O\left(\sqrt{C^* \log n} + \log n\right).$$

**Problem 5.** We again have $n$ experts, each making a binary prediction at each time step. However, we would like to make sure we do well even if the best expert does badly overall, as long as for each "chunk" $I_{t_1,t_2} = \{t_1, t_1+1, \ldots, t_2-1, t_2\}$, we do well compared to the best expert *for this chunk*.

To try and get this, consider the variant of the MWU, where we only penalise an expert by multiplying its weight by $1/2$ *if its current weight is at least* $1/3$ *of the average weight of all experts*.

We want to show that for every $1 \leq t_1 \leq t_2 \leq T$, the maximum number of mistakes $C(t_1, t_2)$ that the algorithm makes over $I_{t_1,t_2}$ is at most $O(C^*(t_1, t_2) + \log n)$, where $C^*(t_1, t_2)$ is the number of mistakes made by the best expert *in that chunk*. (Considering $\beta \in (0, 1)$ to be a constant, e.g., $\beta = 1/2$.)

a) Write down the algorithm.

b) Consider any chunk $I = I_{t_1,t_2}$, and let $t \in I$ be a time step where a mistake is made. Let $W_t$ be the total weight at the beginning of step $t$, and $W_G, W_B, W_L$ be the total weight of (1) experts who made a mistake, (2) experts who did not, and (3) experts who made a mistake but have weight less than $\frac{1}{3} \cdot \frac{W}{n}$. Bound the weight $W_{t+1}$ at the end of step $t$ as a function of $W_G, W_B, W_L, \beta$.

c) Bound the weight $W_{t+1}$ at the end of step $t$ as a function of $W_t, \beta$: show that

$$W_{t+1} \leq \frac{5+\beta}{6} W_t$$

d) Give a lower bound on the weight $w_{i,t_1}$ of *any* expert $i$ at time $t_1$ (start of the chunk). Namely, show that

$$w_{i,t_1} \geq \frac{\beta W_{t_1}}{3n}, \qquad 1 \leq i \leq n$$

e) Letting $W_{t_1}$ the total weight at the beginning of the chunk, and $W_{t_2}$ at the end, show that

$$W_{t_2} \geq \beta^{C^*(t_1,t_2)} \cdot \frac{\beta W_{t_1}}{3n}$$

f) Conclude.

**Solution 5.**

b), c) If we make a mistake at time $t$, the algorithm will have $W_G > W_B$ and thus

$$W_G \geqslant \frac{1}{2}(W_G + W_B) = \frac{1}{2} W_t.$$

And we have $W_L \leqslant \frac{1}{3} W_t$,

$$\begin{aligned}
W_{t+1} &= \beta W_G + W_B + (1-\beta) W_L \\
&= W_G + W_B + (1-\beta)(W_L - W_G) \\
&\leqslant W_t + (1-\beta)\left(\frac{1}{3} W_t - \frac{1}{2} W_t\right) \\
&= W_t - \frac{1-\beta}{6} = \frac{5+\beta}{6}.
\end{aligned}$$

d) Denote by $\tilde{W}$ the total weight when expert $i$ got penalised before $t_1$. Since we only decrease weights as time progress, we have $\tilde{W} \geqslant W_{t_1}$. If $i$ gets penalised and denote its weight at that time $\tilde{w}_i$, it must at the time have

$$\tilde{w}_i \geqslant \frac{\tilde{W}}{3n}.$$

And $w_{i,t_1} = \beta\tilde{w}_i \geqslant \frac{\beta\tilde{W}}{3n} \geqslant \frac{\beta W_{t_1}}{3n}$. If expert $i$ has not been penalised up until $t_1$, we know that $w_{t_1} = 1$ and therefore $w_{t_1} = 1 \geqslant \frac{\beta W_{t_1}}{3n}$.

e) The best expert makes at most $C^{*(t_1,t_2)}$ mistakes; let $j$ be the index of that expert. Then,

$$w_{j,t_2} \geqslant \beta^{C^{*(t_1,t_2)}} w_{j,t_1} \geqslant \beta^{C^{*(t_1,t_2)}} \frac{\beta W_{t_1}}{3n}.$$

Finally,

$$W_{t_2} = \sum_{i=1}^{n} w_{i,t_2} \geqslant w_{j,t_2} \geqslant \beta^{C^{*(t_1,t_2)}} \frac{\beta W_{t_1}}{3n}$$

## Advanced

**Problem 6.** In the setting of the MWU, we have $n$ experts, each making a binary prediction at each time step. Now, assume that we know that, for every $1 \leq k \leq n$, the $k$-th expert makes at most $k$ mistakes.

a) What bound can you show on $C(T)$ when running the MWU algorithm with parameter $\beta$?

b) What bound can you show on $\mathbb{E}[C(T)]$ when running the Randomised MWU algorithm with parameter $\beta$?

**Solution 6.**

a) We can use an analysis very similar to that given in class for the MWU algorithm. On the one hand, if the algorithm makes $C$ mistakes then after these mistakes the total weight $W$ of all experts will be at most $n\left(\frac{1+\beta}{2}\right)^C$. On the other hand, we now know that the $k$-th expert makes at most $k$ mistakes, so the lower bound on the total weight we have is $W \geq \beta + \beta^2 + \cdots + \beta^n = \beta \cdot \frac{1-\beta^n}{1-\beta}$. Solving the inequality

$$\beta \cdot \frac{1-\beta^n}{1-\beta} \leq n\left(\frac{1+\beta}{2}\right)^C$$

we obtain

$$C(T) \leq \frac{\log_2 \frac{1-\beta}{\beta(1-\beta^n)} + \log_2 n}{\log_2 \frac{2}{1+\beta}} = \frac{\ln \frac{1-\beta}{\beta(1-\beta^n)} + \ln n}{\ln \frac{2}{1+\beta}}$$

as our bound on the number of mistakes.

b) The analysis is similar to that of the Randomized MWU algorithm from the lecture. Let $F_i$ denote the fraction of weight at the $i$-th trial on experts giving an incorrect advice, so that $C = \sum_{i=1}^T F_i$. On the one hand, we have that $W$ (the final total weight of all experts) equals $n\prod_{i=1}^T (1-(1-\beta)F_i)$. On the other hand, we know that expert $k$ has weight at least $\beta^k$, so here again $W \geq \beta + \beta^2 + \cdots + \beta^n = \beta \cdot \frac{1-\beta^n}{1-\beta}$. Putting these together as in the lecture,

$$\mathbb{E}[C(T)] = \sum_{i=1}^T F_i \leq \frac{\ln \frac{1-\beta}{\beta(1-\beta^n)} + \ln n}{1-\beta}.$$