Warm-up

Problem 1. Consider a deck of $4n$ cards, with n ♠, n ♡, n ♢, and n ♣. After it is shuffled uniformly at random, what is the expected number of consecutive pairs of the same suit?

Problem 2. A computer randomly generates a 2024-bit long binary string. What is the expected number of consecutive runs of 3 ones? (For instance, the 4-bit binary string 1111 has 2 such consecutive runs, while 0111 only has 1.)

Problem 3. An integer $1 \leq i \leq n$ is called a *fixed point* of a given permutation $\pi\colon \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ if $\pi(i) = i$. Show that the expected number of fixed points of a uniformly randomly chosen permutation π is 1. What is the variance?

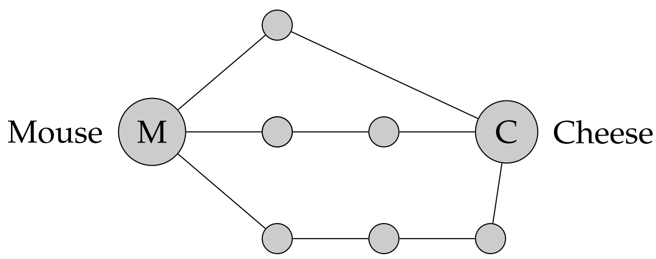
Problem 4. (1) Give a random variable X over $[0,\infty)$ such that $\mathbb{E}[X] = \infty$. (2) Give a random variable X over $\mathbb{N}$ such that $\mathbb{E}[X] = \infty$.

Problem 5. Prove the fact from the lecture: if X has a finite variance, then $\operatorname{Var} X = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Problem solving

Problem 6. Prove the fact from the lecture: if X takes values in $\mathbb{N} = \{0,1,2,\dots,\}$ and $\mathbb{E}[X]$ is finite, then $\mathbb{E}[X] = \sum_{n=1}^{\infty} \Pr[\, X \geq n \,]$.

Problem 7. Consider the following map: each edge represents a path (of length one) between two different locations. To reach the cheese, the mouse needs to take a path connecting locations M and C.



Unfortunately, cats have heard of this plan, and will try to intercept the mouse. These cats are not the brightest, thankfully, and behave randomly: namely, each edge will be occupied by a cat, independently of all other edges, with some fixed probability $p \in (0,1)$. The mouse cannot go on any edge that has a cat, of course. (Once the cats have randomly decided their position at the beginning, they stay there once and for all, effectively "killing" that edge as far as the mouse is concerned.)

a) Give the probability that the mouse still has a path leading to the cheese.

b) Give the probability that the mouse still has a path of length at most 3 leading to the cheese.

c) Give the expected numbers of cats on the map.

Problem 8. Let A be an array of n distinct numbers. We say that an index $1 \leq i \leq n$ is "prefix-maximum" if $A[i]$ is the biggest number so far, that is, if $A[j] < A[i]$ for all $j < i$. Let $\mathrm{pf}(A)$ denote the number of prefix-maximum indices of A.

a) What is $\mathrm{pf}(A)$ if A is sorted (increasing)?

b) Suppose that we permute the elements of A uniformly at random to get an array B. Show that
$$\mathbb{E}[\mathrm{pf}(B)] = H_n = O(\log n)\,,$$
where $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$ is the n-th Harmonic number.

Advanced

Problem 9. Given two values $x, y \in \{0, 1\}$, their XOR $x \oplus y$ is equal to their sum modulo 2, or equivalently, is 1 if $x + y$ is odd, and 0 otherwise. This generalises to n bits as follows: for $x_1, \ldots, x_n \in \{0, 1\}$,

$$x_1 \oplus x_2 \oplus \cdots \oplus x_n = \begin{cases} 0 \text{ if } \sum_{i=1}^n x_i \text{ is even} \\ 1 \text{ if } \sum_{i=1}^n x_i \text{ is odd} \end{cases}$$

Suppose that $X_1, \ldots, X_n, \ldots$ are independent Bernoulli random variables with parameter $p \in [0, 1]$, and, for any $n \geq 1$, let $Y_n = X_1 \oplus X_2 \oplus \cdots \oplus X_n$. This is itself a Bernoulli random variable: let's call its parameter p_n.

a) Compute the first few values of p_n when $p = 1/2$, $p = 0$, and $p = 1$. Establish the expression of p_n (as a function of n) for these particular cases. Interpret the result.

b) In general, as a function of p, what is p_0? p_1? p_2?

c) Give a recurrence relation for p_n.

d) Solve the recurrence to obtain the expression for p_n. Show that it always converge to $1/2$. How fast?