COMPx270: Randomised and Advanced Algorithms

Lecture 5: Graph algorithms

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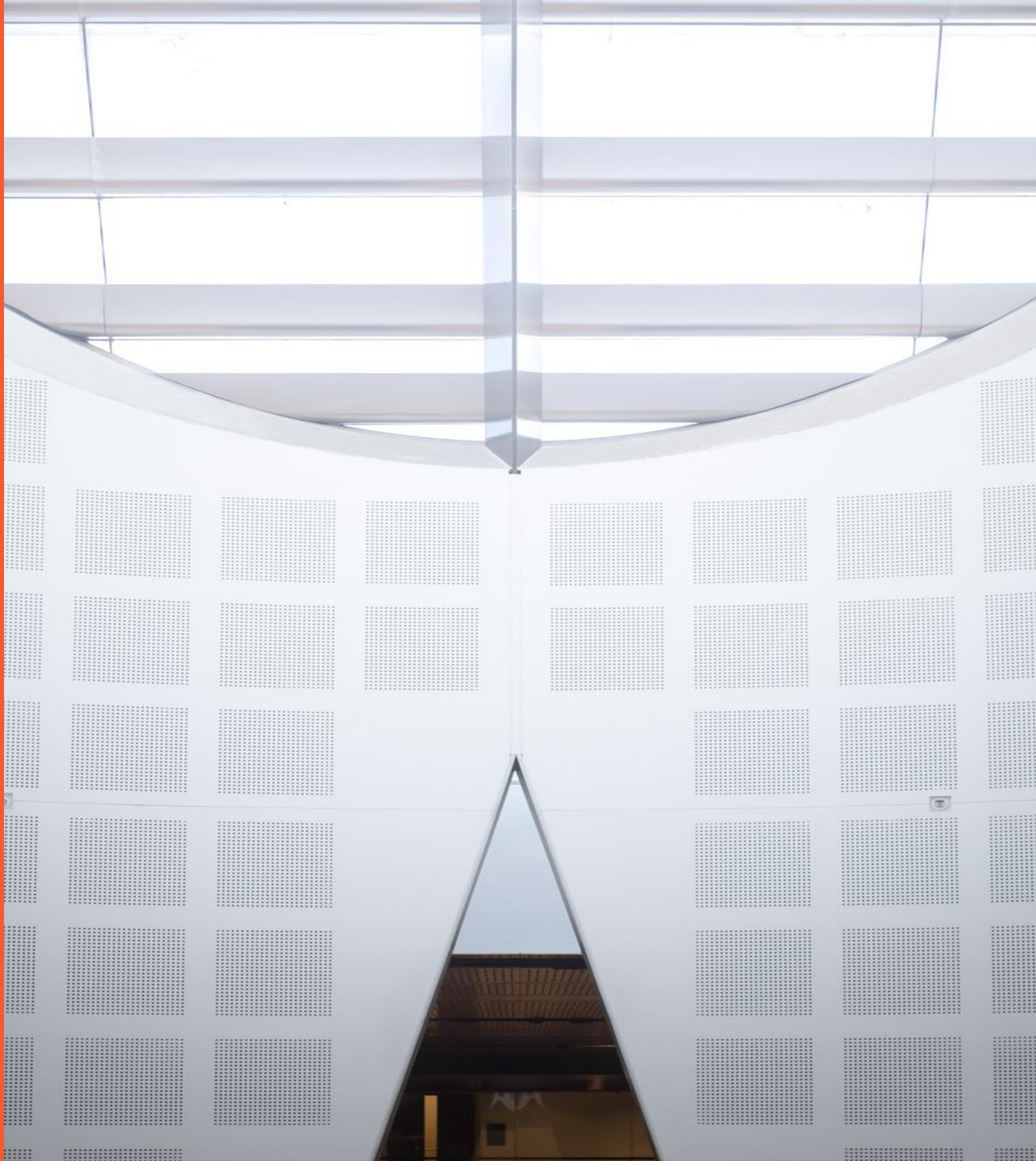
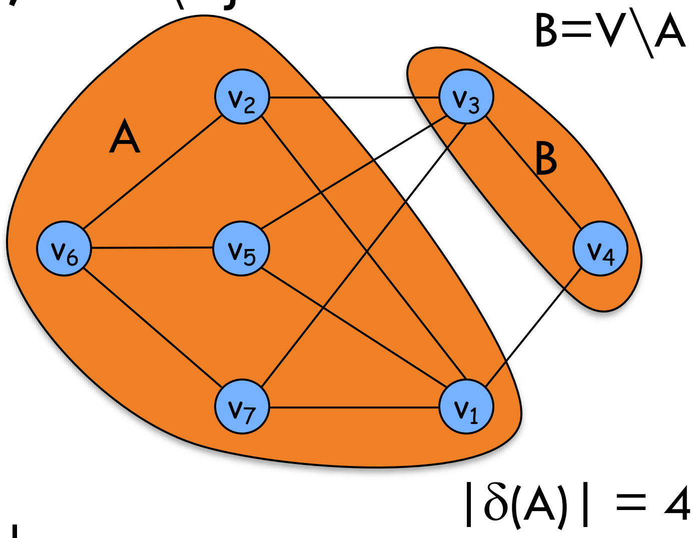


Global Minimum Cut

Input: A connected, undirected graph $G = (V, E)$.

For a set $A \subset V$ let $\delta(A) = \{(u,v) \in E : u \in A,\ v \in V \setminus A\}$.



Aim: Find a cut (A, B) minimizing $|\delta(A)|$.

Global Minimum Cut

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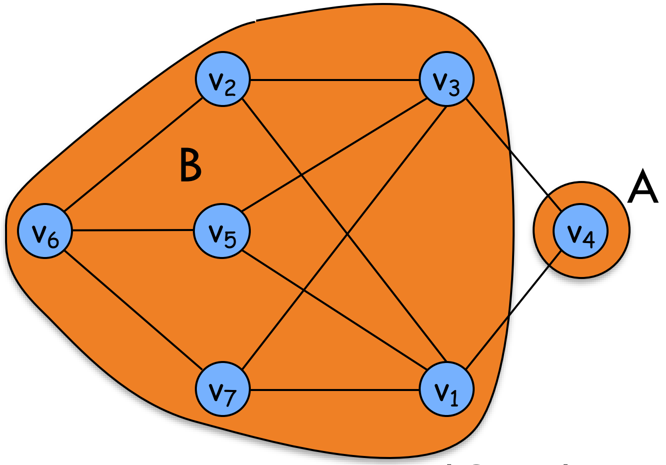


$|\delta(A)| = 2$

Aim: Find a cut (A, B) of minimum cardinality.

Global Minimum Cut

Applications: Partitioning items in a database, identifying clusters of related documents, network reliability, network design, circuit design, TSP solvers.

Network flow solution.

- Replace every edge (u, v) with two directed edges (u, v) and (v, u).
- Pick some vertex s and compute min s-v cut separating s from each other vertex $v \in V$.

Running time: $O((n-1) \cdot \text{MaxFlows})$

Global Minimum Cut

F = value of max flow

Max-Flow:

Running time: $O((n-1)\cdot \text{MaxFlows})$

Ford-Fulkerson: $O(mF)$

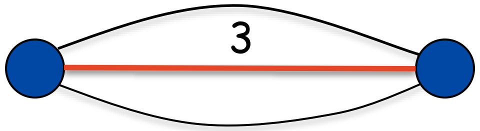
______* $O(m \log F)$

Edmonds-Karp $O(mn)$

Best? $O\left(m \log \frac{n^2}{m}\right)$

Karger's Contraction Algorithm

Definition: A multigraph is a graph that allows multiple edges between a pair of vertices.



Karger's Contraction Algorithm

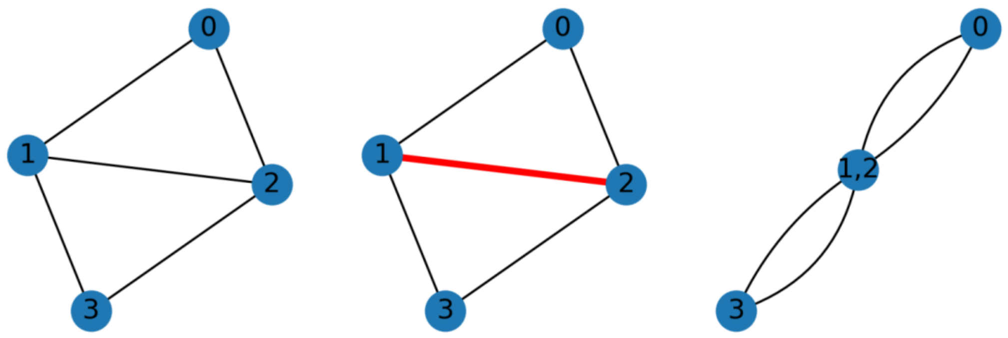
Definition: A multigraph is a graph that allows multiple edges between a pair of vertices.

Karger's Contraction Algorithm

Let G=(V,E) be a multigraph (without self-loops).

Contraction of an edge $e=(u,v) \in E \quad \Rightarrow \quad G \backslash e$

- Replace u and v by single new super-node w
- Replace all edges (u,x) or (v,x) with an edge (w,x)
- Remove self-loops to w.



Karger's Contraction Algorithm

Require: multigraph $G = (V, E)$

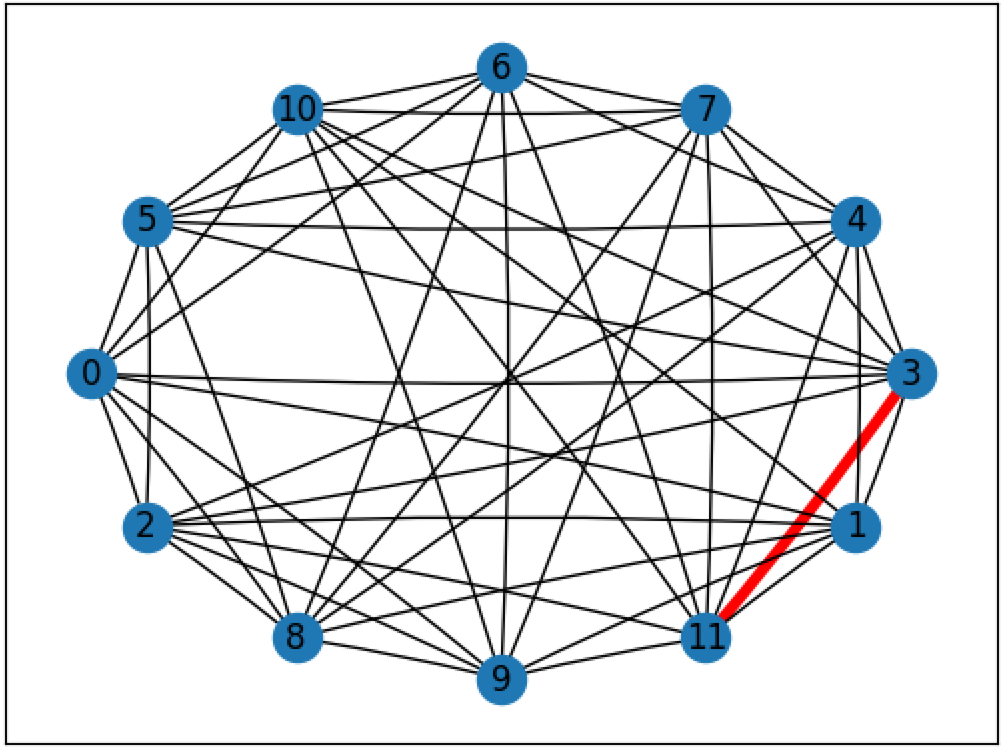
while $|V| > 2$ **do**

  Pick an edge $e \in E$ uniformly at random

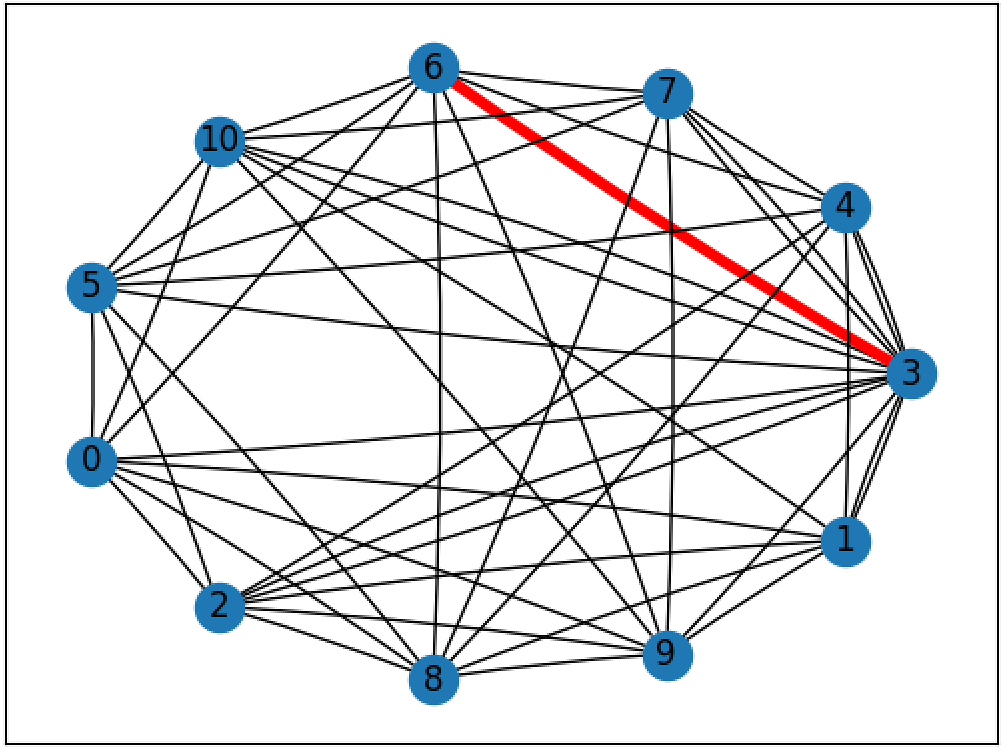
  Contract it, and let $G \leftarrow G/e$

return the cut defined by the remaining two vertices.

Karger's contraction algorithm



Karger's contraction algorithm



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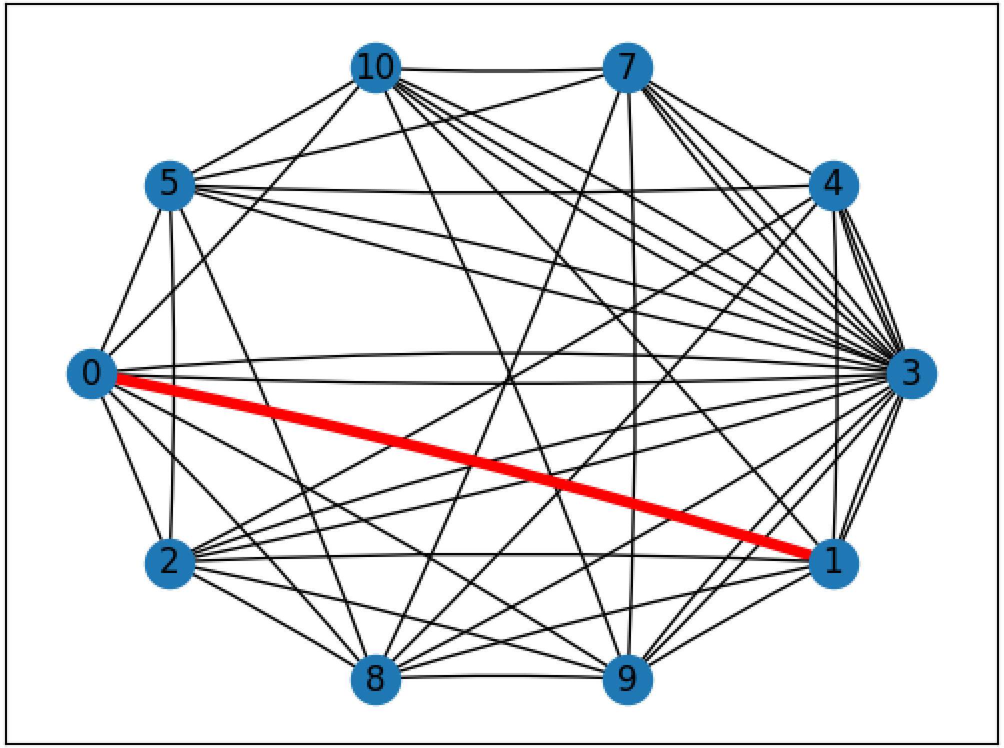

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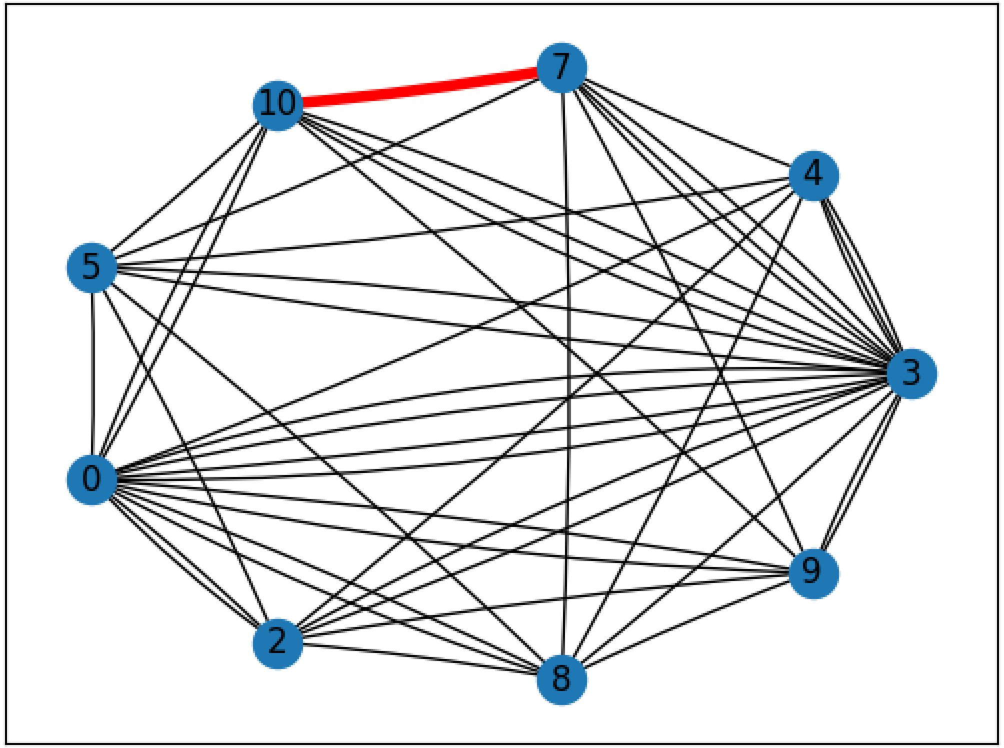

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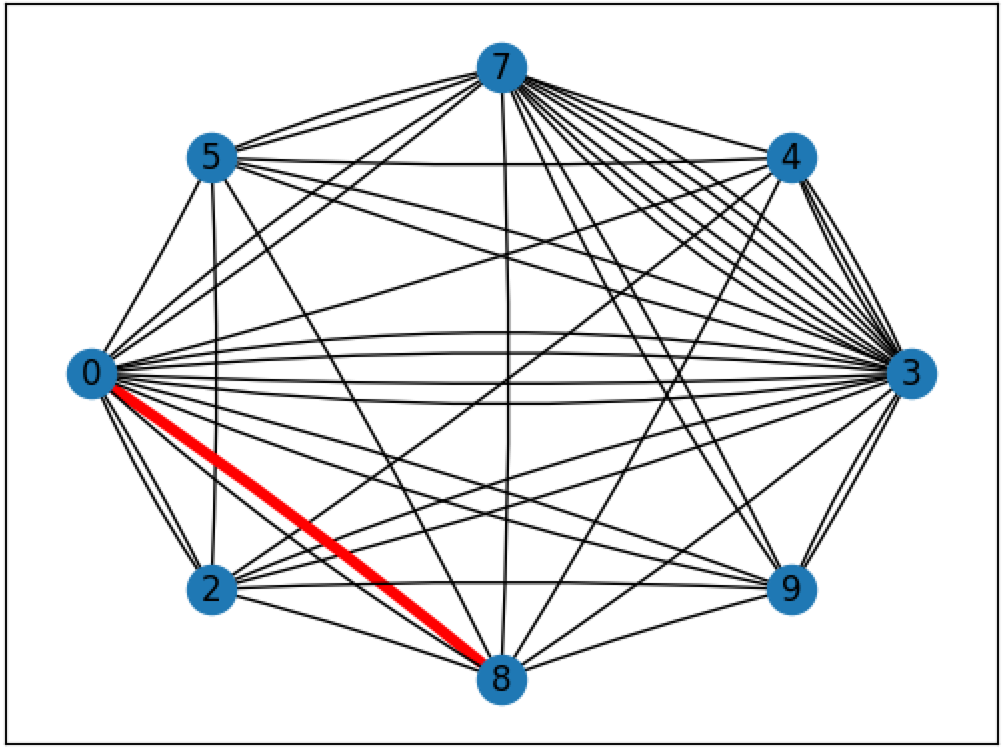

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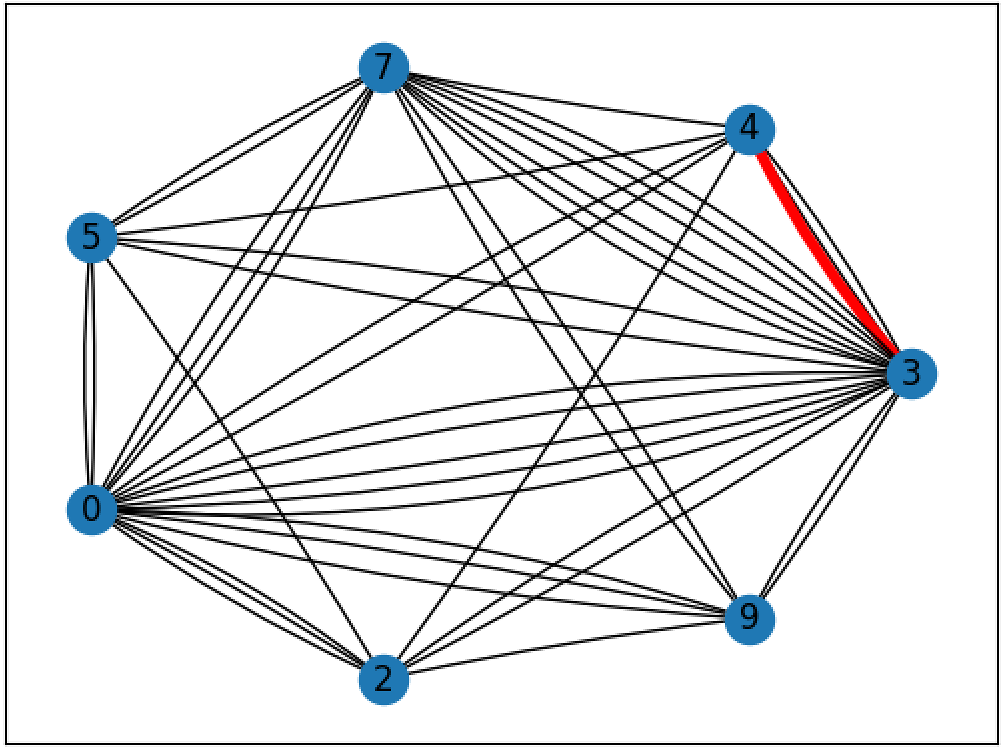

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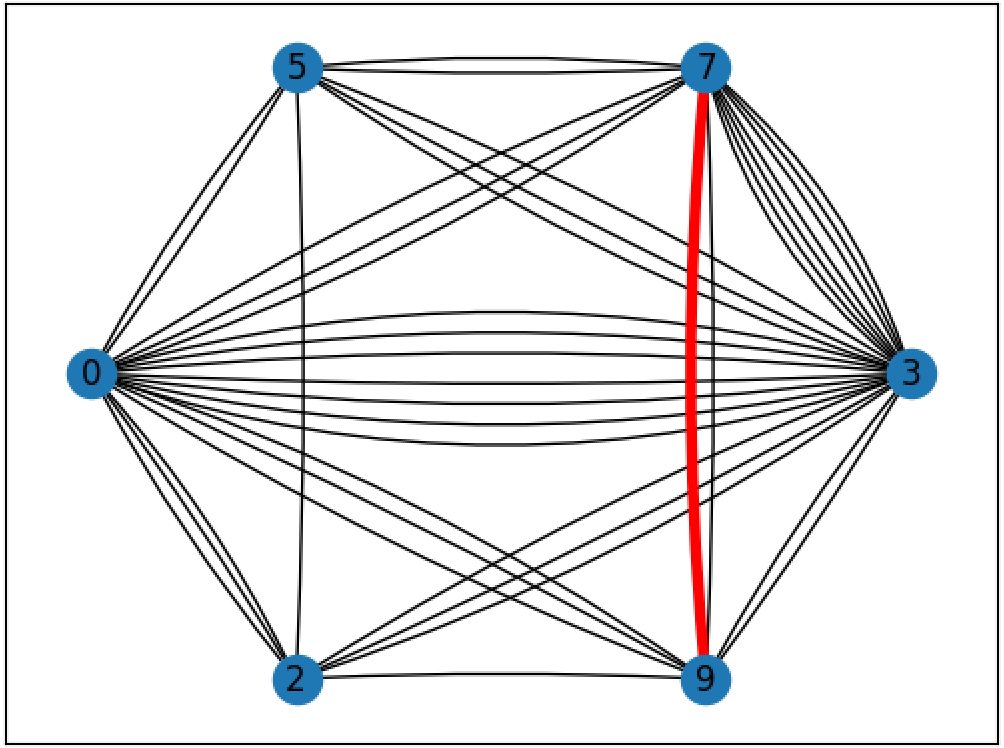

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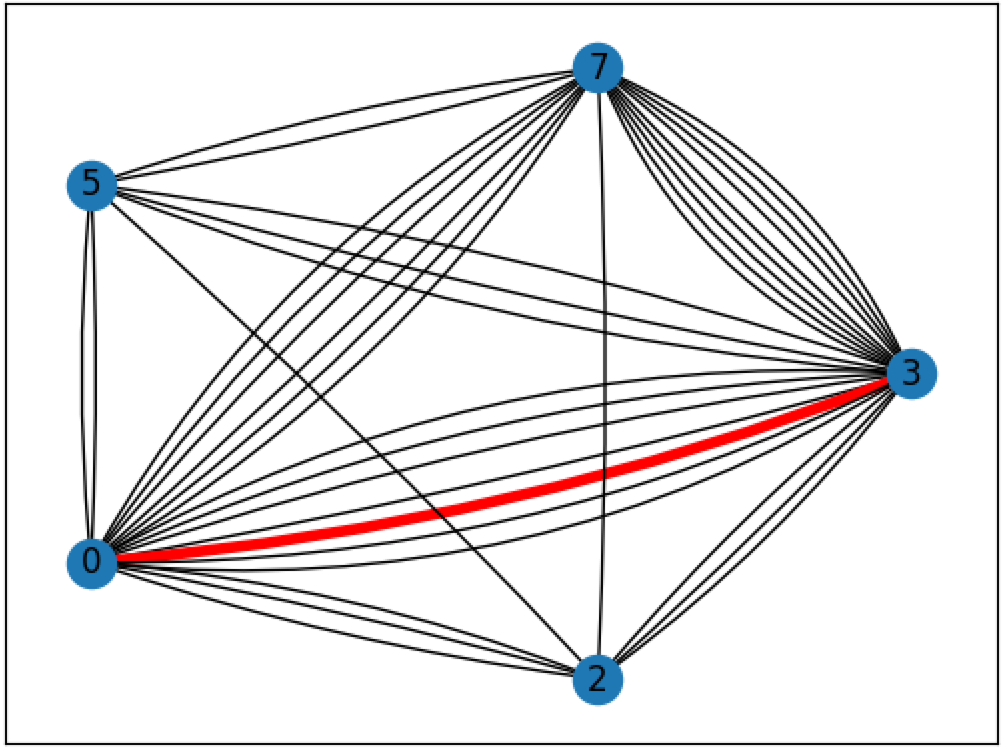
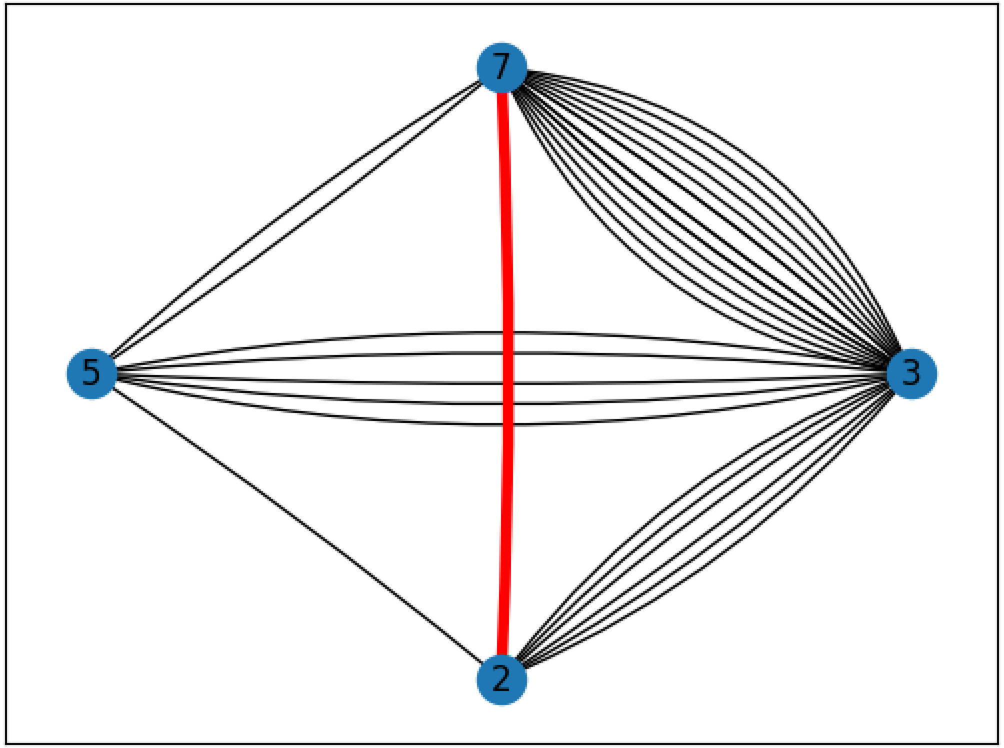
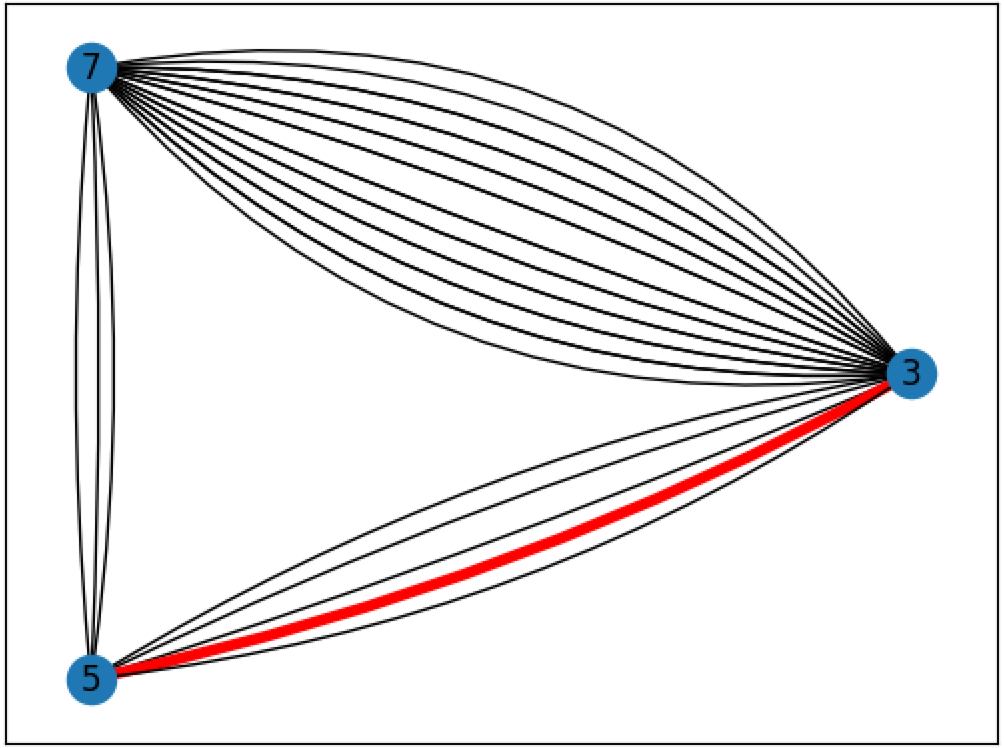


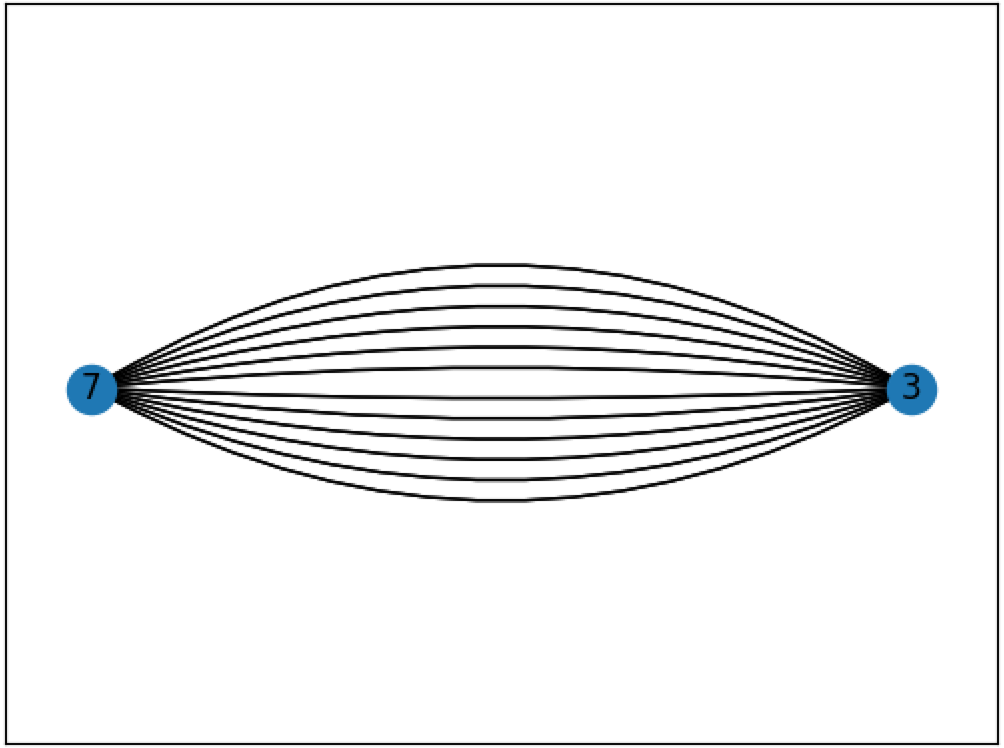
Karger's contraction algorithm



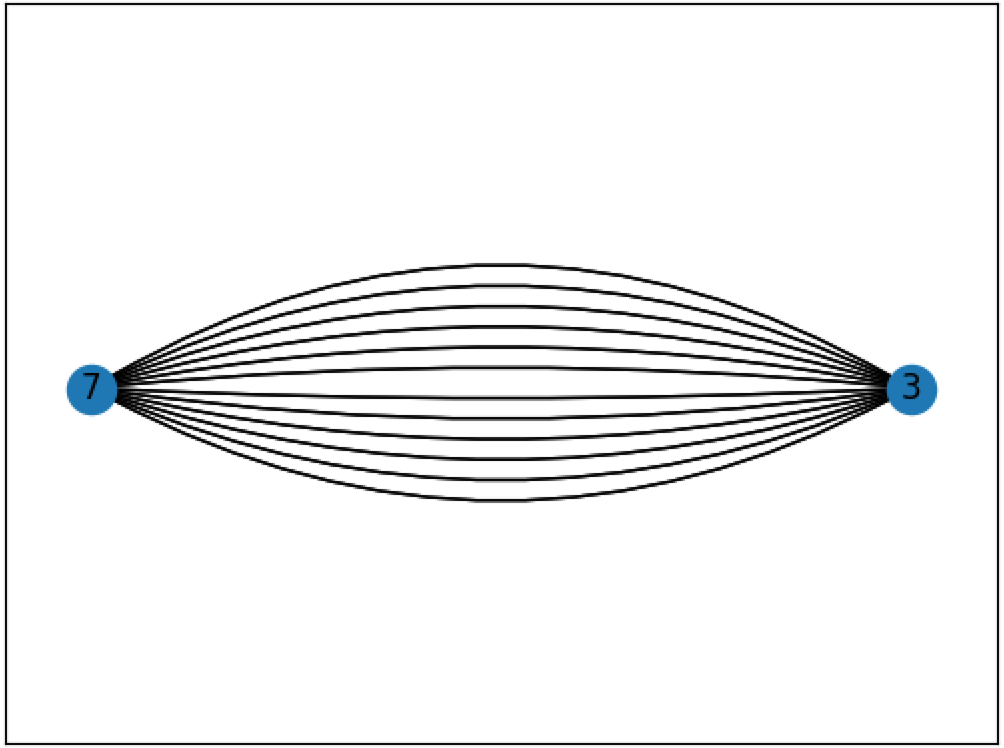
Karger's contraction algorithm



Karger's contraction algorithm



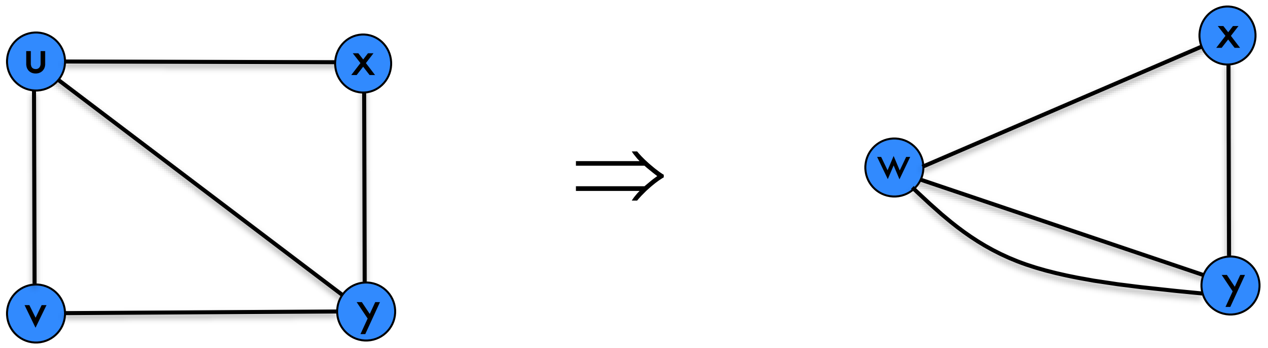
Karger’s contraction algorithm



The End.

Karger’s Contraction Algorithm

Observation: An edge (u,v) contraction preserves the cuts (A,B) where u and v are both in A or both in B.



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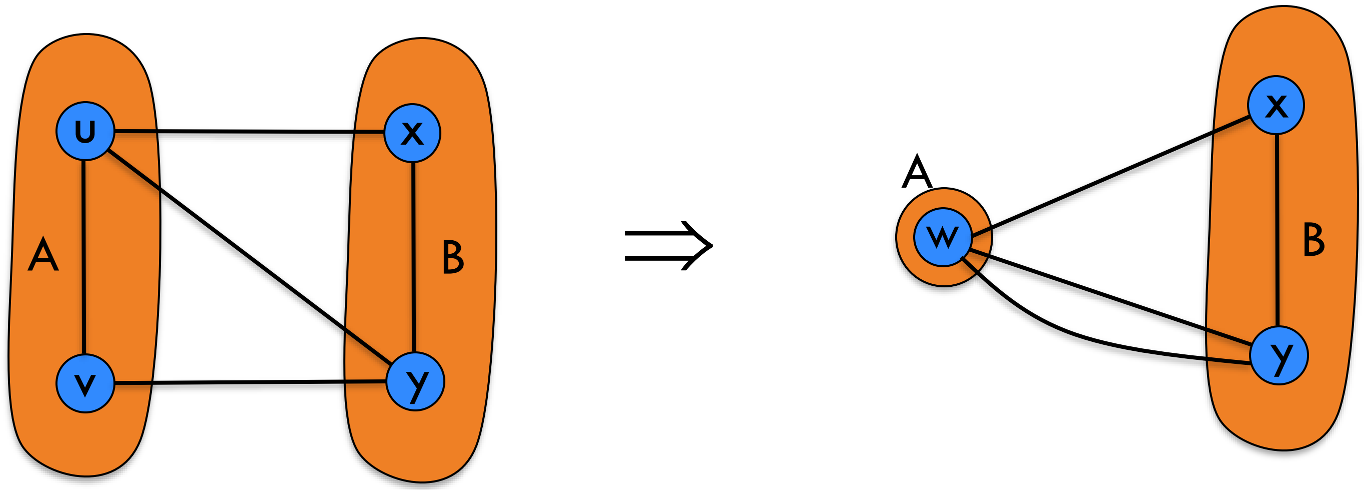


Karger's Contraction Algorithm

Observation: An edge (u,v) contraction preserves the cuts (A,B) where u and v are both in A or both in B.

$\boxed{\text{If } u,v \in A \text{ then } \delta_G(A) = \delta_{G \backslash e}(A). \text{ (with u and v replaced with "uv")}}$

Karger's Contraction Algorithm

Observation: If (A,B) is a minimum cut, then we are less likely to choose an edge (u,v) crossing it!

Karger's Contraction Algorithm

Require: multigraph $G = (V, E)$

1: **while** $|V| > 2$ **do**

2: Pick an edge $e \in E$ uniformly at random

3: Contract it, and let $G \leftarrow G/e$

4: **return** the cut defined by the remaining two vertices.

Claim: This algorithm has a reasonable chance of finding a min cut.

Prove the claim

Claim: If C is a min-cut, then the algorithm returns it with probability at least $2/n^2$.

Prove the claim

Claim: If C is a min-cut, then the algorithm returns it with probability at least $2/n^2$.

E_i = "C survives i-th step"

Proof.

$$\Pr[C \text{ survives}] = \Pr[C \text{ survives all } n-2 \text{ steps}] = \Pr[E_1 \cap E_2 \cap \dots \cap E_{n-2}]$$

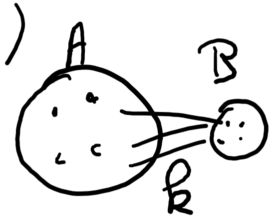
$$= \Pr[E_1] \cdot \Pr[E_2 \mid E_1] \cdot \Pr[E_3 \mid E_1 \cap E_2] \cdots \Pr[E_{n-2} \mid E_1 \cap \dots \cap E_{n-3}]$$

$$\Pr[E_{i+1} \mid E_1 \cap \dots \cap E_i] \geq ? \qquad (0 \leq i \leq n-3)$$

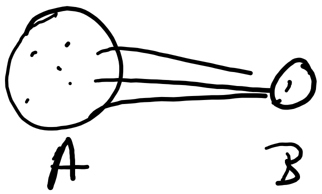
$k = |C|$

$$\Pr[E_1] = 1 - \frac{k}{m}$$

$C = (A, B)$



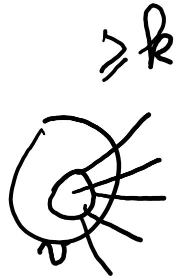
At step i $\quad G_i = (V_i, E_i)$ $\quad |V_i| = n - i$ ⊛



$E_{i+1} \mid E_1 \cap \dots \cap E_i$ $\quad \Pr[E_{i+1} \mid E_1 \cap \dots \cap E_i] = 1 - \frac{k}{|E_i|}$

$$\geq 1 - \frac{2k}{k(n-i)} \quad (\ddagger)$$

$$= 1 - \frac{2}{n-i} \quad ☆$$



☆ $\boxed{\text{Handshaking Lemma}}$

$$2|E_i| = \sum_{v \in V_i} \underbrace{\deg_v}_{\geq k} \geq k(n-i) \quad (\ddagger)$$

$|V_i| = n-i$ terms ⊛

$$\Pr[E_1 \cap \dots \cap E_{n-2}] \geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right)$$

$$= \prod_{i=0}^{n-3} \frac{n-i-2}{n-i} \underset{j=n-i}{=} \prod_{j=3}^{n} \frac{j-2}{j} = \frac{\prod_{j=3}^{n} (j-2)}{\prod_{j=3}^{n} j}$$

$$= \frac{(n-2)!}{\frac{n!}{2 \cdot 1}} = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}} \geq \boxed{\frac{2}{n^2}}$$

Amplification

To amplify the probability of success, run the contraction algorithm many times.

Require: multigraph $G = (V, E)$, integer T

1: **for** $1 \le t \le T$ **do** ▷ Use fresh (independent) random bits for each

2: Run Algorithm on G, let C_t be the output

3: **return** the smallest cut among all cuts $C_1, \ldots, C_T$ obtained

---

Amplification

To amplify the probability of success, run the contraction algorithm many times.

Claim: If we repeat the contraction algorithm $r\binom{n}{2}$ times with independent random choices, the probability that all runs fail is at most $(1/e)^r$.

WANT

$$\Pr[\text{all } T \text{ runs are "unlucky"}] = \left(1 - \Pr[\text{one run is lucky}]\right)^T \le \left(1 - \frac{2}{n^2}\right)^T \le \delta$$

$$(1-x)^T \le e^{-xT}$$

$$\left(1 - \frac{2}{n^2}\right)^T \le e^{-\frac{2T}{n^2}}$$

Suffices: $e^{-\frac{2T}{n^2}} \le \delta$

$$-\frac{2T}{n^2} \le \ln \delta$$

$\boxed{T \ge \frac{n^2}{2} \ln \frac{1}{\delta} \text{ suffices}}$

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Running time?

Karger's Contraction Algorithm

Require: multigraph $G = (V, E)$

n-2 times

while $|V| > 2$ **do**

  Pick an edge $e \in E$ uniformly at random

  Contract it, and let $G \leftarrow G/e$

Takes time O(n)

return the cut defined by the remaining two vertices.

Running time?

The algorithm is iterated $O(n^2 \log n)$ times…total running time $O(n^4 \log n)$.

Karger's Contraction Algorithm

Can we do better?

Yes

At step i, we "kill" that fixed C w.p $\frac{2}{n-i}$

$\Pr\left[C \text{ survives } (s) \text{ steps}\right] = \frac{s(s-1)}{n(n-1)}$

$\approx \frac{s^2}{n^2}$

$\left(\text{Set } s = \frac{n}{\sqrt{2}}\right)$

$= \frac{1}{2}$

Remains:

$\approx n\left(1 - \frac{1}{\sqrt{2}}\right)$ vertices

Improved algorithm

Improvement. [Karger-Stein 1996]

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n/\sqrt{2}$ nodes remain.
- Run contraction algorithm until $n/\sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph, and return best of two cuts.

Improved algorithm

procedure ModifiedKarger($G = (V, E), s$)
    while $|V| > s$ **do**
        Pick an edge $e \in E$ uniformly at random
        Contract it, and let $G \leftarrow G/e$
    return G
procedure KargerStein($G = (V, E)$)
    if $|V| \leq 6$ **then**
        return a minimum cut   ▷ Brute-force computation
    Set $s \leftarrow \lceil n/\sqrt{2} + 1 \rceil$ $\frac{n}{\sqrt{2}}$
    ▷ **Contraction**
        $G_1 \leftarrow$ ModifiedKarger(G, s)   ← C still alive w.p. $\geq 50\%$
        $G_2 \leftarrow$ ModifiedKarger(G, s)   ← C still ——— $\geq 50\%$
    ▷ **Recursion**
        $C_1 \leftarrow$ KargerStein(G_1)
        $C_2 \leftarrow$ KargerStein(G_2)
    return the smallest cut among C_1, C_2

Improved algorithm: Karger-Stein

procedure ModifiedKarger($G = (V, E), s$)
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Running time?

Improved algorithm: Karger-Stein

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$O(n^2)$

recursive

$$T(n) = 2\,T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$$

$$= (\ldots)$$

$$= O(n^2 \log n)$$

Good.. if probability of success $\gg \frac{1}{n^2}$

Improved algorithm: Karger-Stein

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  return the smallest cut among C_1, C_2

Success probability?

Improved algorithm: Karger-Stein

```
1:  procedure ModifiedKarger(G = (V, E), s)
2:      while |V| > s do
3:          Pick an edge e ∈ E uniformly at random
4:          Contract it, and let G ← G/e
5:      return G
6:  procedure KargerStein(G = (V, E))
7:      if |V| ≤ 6 then
8:          return a minimum cut                ▷ Brute-force computation
9:      Set s ← ⌈n/√2 + 1⌉
10:     ▷ Contraction
11:         G1 ← ModifiedKarger(G, s)
12:         G2 ← ModifiedKarger(G, s)
13:     ▷ Recursion
14:         C1 ← KargerStein(G1)
15:         C2 ← KargerStein(G2)
16:     return the smallest cut among C1, C2
```

$p(n)$

Success probability?

G_1 still has C

$$\Pr[C_1 \text{ is a min cut}] \geq \frac{1}{2} \cdot p\left(\frac{n}{\sqrt{2}}\right)$$

probability line 14 is successful

$$\Pr[C_2 \text{ is a min cut}] \geq \frac{1}{2} \cdot p\left(\frac{n}{\sqrt{2}}\right)$$

$$1 - \Pr[C_2 \text{ min cut}] \leq 1 - \frac{1}{2} p\left(\frac{n}{\sqrt{2}}\right)$$

$$p(n) = 1 - \Pr[C_1, C_2 \text{ are not min cuts}] = 1 - \Pr[C_1 \text{ not min cut}] \underbrace{\Pr[C_2 \text{ not min cut}]}$$

$$\geq 1 - \left(1 - \frac{1}{2} p\left(\frac{n}{\sqrt{2}}\right)\right)^2$$

$$\boxed{p(n) \geq 1 - \left(1 - \frac{1}{2} p\left(\frac{n}{\sqrt{2}}\right)\right)^2}$$

Claim $\quad p(n) \geq \dfrac{1}{2\log n + 2} = \Omega\left(\dfrac{1}{\log n}\right)$

Improved algorithm: Karger-Stein

Theorem. [Karger-Stein 1996] The Karger-Stein algorithm runs in time $O(n^2 \log n)$ and returns a min cut with probability at least $\Omega(1/\log n)$.

Corollary. The "best-of-T" Karger-Stein algorithm runs in time $O(n^2 \log^2 n)$ and returns a min cut with probability at least 99%.

$\left(T = O(\log n) \right.$ solve: $\left. (1 - p(n))^T \leq \frac{1}{100} \right)$

"Proof" $\quad p(n) \geq 1 - \left(1 - \frac{1}{2} p\left(\frac{n}{\sqrt{2}}\right)\right)^2$

Set $f(t) = p(\sqrt{2}^t)$ $\qquad n = \sqrt{2}^t \qquad t = 2\log n$

$f(t) \geq 1 - \left(1 - \frac{1}{2} f(t-1)\right)^2$

Set $g(t) = \frac{4}{f(t)} - 1$ $\qquad \left(f(t) = \frac{4}{g(t)+1} \right)$ $\qquad \frac{4}{g(t)+1} \geq 1 - \left(1 - \frac{2}{g(t-1)+1}\right)^2$

Improved algorithm: Karger-Stein

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Best known. [Karger 2000] $O(m \log^3 n)$.

And now, for something completely different

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Theorem. An undirected graph G=(V,E) has at most ________ distinct min cuts.

And now, for something completely different?

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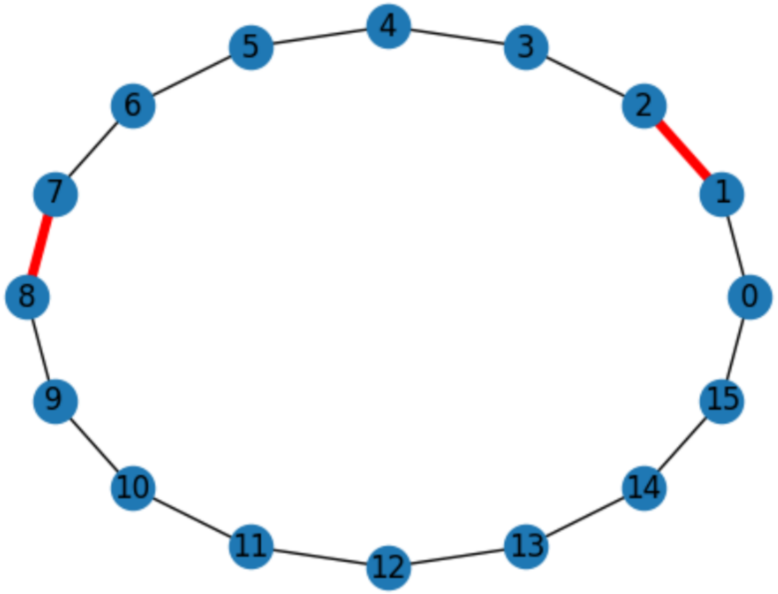
And now, for something completely different?

Theorem. An undirected graph G=(V,E) has at most $\frac{n(n-1)}{2}$ distinct min cuts. And this is tight.



And now, for something completely different?

Theorem. An undirected graph G=(V,E) has at most $\frac{n(n-1)}{2}$ distinct min cuts.

Proof. Fix C, any fixed min. cut.

$$\Pr[\text{Karger's algo output this } C] \geq \frac{2}{n(n-1)}$$

So

$$1 \geq \Pr[\text{algo output some min-cut}] = \sum_{C:\text{ min cut}} \Pr[\text{output this } C] \geq (\#\text{ min cuts}) \cdot \frac{2}{n(n-1)}$$

$\square$