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COMPx270: Randomised and Advanced Algorithms
Lecture 4: Derandomisation

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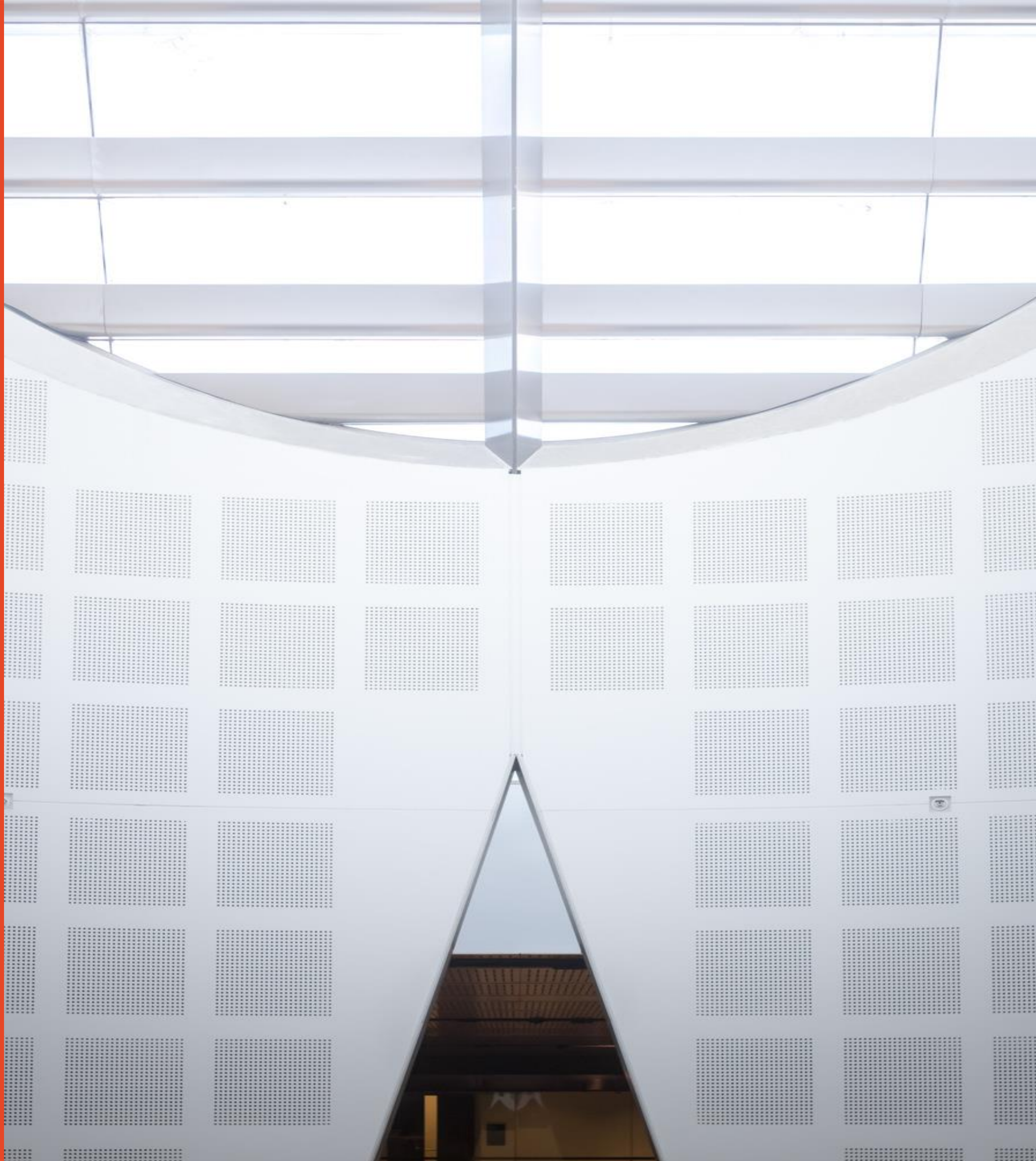
THE UNIVERSITY OF SYDNEY



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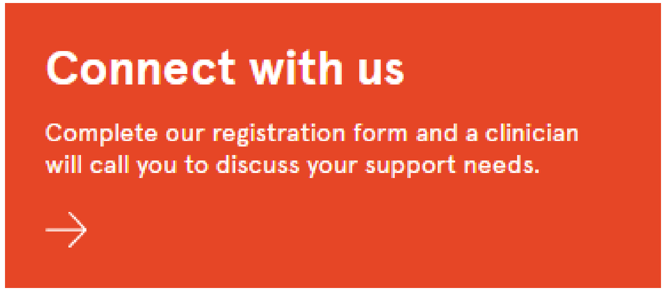

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An announcement (or three)

- HW1, Problem 5: β should be 2β (updated assignment tonight)
- Office Hours (OH) this **Friday, 3:30-5pm, J12 302 + Zoom**
- Simple extensions (you have them by default!)

A question

You have a randomised algorithm A which runs in time T(n) and solves task X (say, decision problem) with probability .99. Is there a deterministic algorithm B which solves X and runs in time...

- O(T(n))
- poly(T(n))
- exp(T(n))
- No/we don't know

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You have a randomised algorithm A which runs in time T(n) and solves task X (say, decision problem) with probability .99. Is there a deterministic algorithm B which solves X and runs in time...

- O(T(n)) ?
- poly(T(n)) ?
- exp(T(n)) ✓
- No/we don't know

An answer?

That's **complicated**. This is what derandomization asks, and there is a lot of work on this: one of the major unsolved question in theoretical computer science.

P v. BPP

↑

class of decision problems w/ a poly-time Monte Carlo Algo

Let's not stop here though

We know how to derandomize **some** algorithms, and there are **some** general techniques.

Method 1: PRNG 🎲

The goal is to reduce the amount of randomness required, by generating a lot of "good enough" pseudorandom bits: good enough to fool the algorithm.

$G: \{0,1\}^\ell \to \{0,1\}^n \qquad n \gg \ell$

$\forall A \in \mathcal{A}$

$TV(A(G(U_\ell)), A(U_n)) \leq \varepsilon$

statistical distance

FACT Under "plausible assumption", PRNGs for $\mathcal{A} = \{$poly time algos$\}$ exist for $\ell = O(\log n)$

Method 1: PRNG 🎲

Why is that useful?

- Random bits don't grow on trees!
- Derandomisation (method 2)

Why is this bad?

- Conditional (under assumptions)

Method 2: Brute force 💪

If the algorithm uses a small number of random seeds, check 'em all.

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```
Require: Input x
for all r ∈ {0,1}^R do
    y ← A(x;r)                        ▷ Run A on x with randomness r
    if V(x,y) = 1 then                ▷ Verify if y is a good solution
        return y                      ▷ If so, we are done
```

T_A

T_V

$$O(2^R(T_A + T_V))$$

Details.

Method 2: Brute force 💪

What if verifying is hard?

① $\Pr[A \text{ is correct}] > 0$

② I can verify. (implies $\geq \frac{1}{2^R}$)

Method 2: Brute force 💪

$A(x) \in \{\text{yes}, \text{no}\}$

What if verifying is hard?

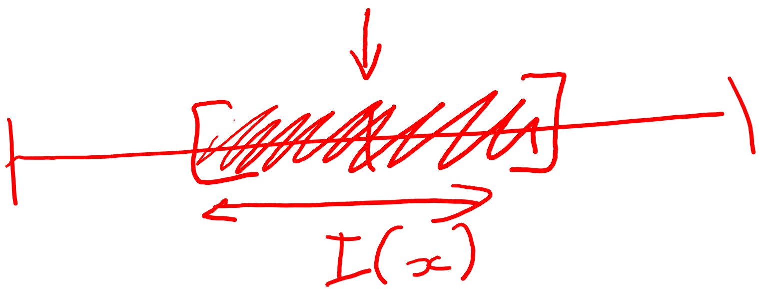
$\Pr[A \text{ is correct}] \geq \frac{2}{3}$

$A(x; R)$

- Majority vote!
- Median trick!

"good"

$\Pr[A(x) \in I(x)] \geq \frac{2}{3}$



Method 2: Brute force 💪

What if the algorithm does not use a small number of random bits?

Method 2: Brute force 💪

What if the algorithm does not use a small number of random bits?

Well, these PRNGs can come in handy…

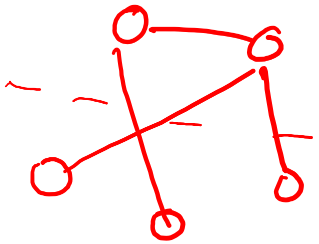
Method 2: Brute force 💪

What if the algorithm does not use a small number of random bits?

Or (sometimes) we can reduce the randomness by carefully looking at the proof.

Method 2: Brute force 💪 via pairwise independence

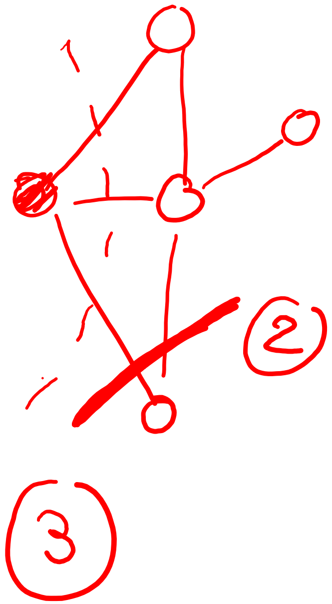
Derandomizing Max-Cut



MAX-CUT: Given an (undirected) graph $G = (V, E)$ on n vertices and m edges, output a cut (A, B) (partition of V) *maximising* the number $c(A, B)$ of edges between A and B.

(It's NP-Hard)

Method 2: Brute force 💪 via pairwise independence



Derandomizing Max-Cut

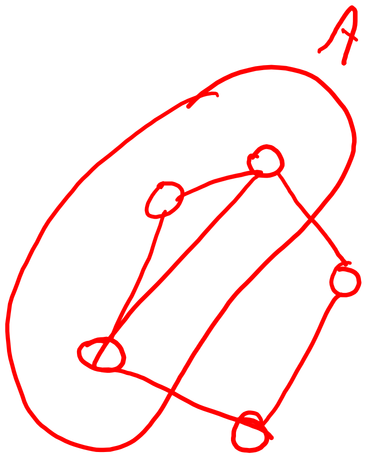
> MAX-CUT: Given an (undirected) graph $G = (V, E)$ on n vertices and m edges, output a cut (A, B) (partition of V) *maximising* the number $c(A, B)$ of edges between A and B.

But we can get a ½-approximation!

Method 2: Brute force 💪 via pairwise independence

$(A, B) \leftarrow (\emptyset, \emptyset)$
for all $v \in V$ **do**
    $X_v \leftarrow \mathrm{Bern}(1/2)$ ▷ Independent of previous choices
    if $X_v = 1$ **then** add v to A
    else add v to B
return (A, B)

Method 2: Brute force 💪 via pairwise independence



$c(A,B) = 2$

Theorem.

$$\mathbb{E}[c(A,B)] \geq \frac{1}{2} m \geq \frac{1}{2}\, \mathrm{OPT}(G)\,.$$

Proof. For $e \in E$, $\mathbb{1}_{e \in C}$ indicates if $e \in$ cut

$e = (u,v)$

$$\mathbb{E}[c(A,B)] = \sum_{e \in E} \mathbb{E}[\mathbb{1}_{e \in C}] = \sum_{e \in E} \Pr[e \text{ is "cut"}] = \sum_{e=(u,v)} \Pr[u \in A, v \in B \text{ or } u \in B, v \in A]$$

$$= \sum_{e=(u,v)} \left(\Pr[u \in A, v \in B] + \Pr[u \in B, v \in A] \right)$$

$\Pr[u \in A, v \in B] = \Pr[u \in A] \Pr[v \in B] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$$= |E| \left(\frac{1}{4} + \frac{1}{4} \right) = \frac{m}{2}$$

Method 2: Brute force 💪 via pairwise independence

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$X_v = h(v)$

Overkill!

Need: $\forall\, u \neq v$; X_v, X_u indep't

Method 2: Pairwise independent hash functions

$|\{h: \mathcal{X} \to \mathcal{Y}\}| = |\mathcal{Y}|^{|\mathcal{X}|}$

Definition 22.1. A family of functions $\mathcal{H} \subseteq \{h\colon \mathcal{X} \to \mathcal{Y}\}$ is a *family of pairwise independent hash functions*, or a *strongly universal hash family*, if, for every $x, x' \in \mathcal{X}$ with $x \neq x'$ and every $y, y' \in \mathcal{Y}$,

$$\Pr_{h \sim \mathcal{H}}\left[h(x) = y, h(x') = y'\right] = \frac{1}{|\mathcal{Y}|^2}$$

where the probability is over the uniformly random choice of $h \in \mathcal{H}$.

want this to be small

Method 2: Pairwise independent hash functions

Fact. Small families of pairwise independent hash functions exist.

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There exist $\mathcal{H} \subseteq \{ \{1, \ldots, n\} \to \{0,1\} \}$ strongly u. hash family

s.t. $\log |\mathcal{H}| = O(\log n)$

Method 2: Pairwise independent hash functions

logn + O(1)

Fact. Small families of pairwise independent hash functions exist.

Proof of derandomization claim.

Pick $h \sim \mathcal{H}$ u.a.r. (O(log n) bits)

1: $(A, B) \leftarrow (\emptyset, \emptyset)$
2: **for all** $v \in V$ **do**
3: $X_v \leftarrow$ Bern(1/2) [h(v)] ▷ Independent of previous choices
4: **if** $X_v = 1$ **then** add v to A
5: **else** add v to B
6: **return** (A, B)

$$\mathbb{E}[c(A,B)] = \sum_{\substack{e \in E \\ e=(u,v)}} \left(\Pr[u \in A, v \in B] + \Pr[u \in B, v \in A] \right)$$

$$= \sum_{e=(u,v)} \left(\underbrace{\Pr[h(u)=1, h(v)=0]}_{\frac{1}{4} \text{ by pairwise ind}} + \underbrace{\cdots}_{\frac{1}{4}} \right)$$

$$= \frac{m}{2}$$

Go over every $h \in \mathcal{H}$ → $O(|\mathcal{H}|) = O(n)$ time.
For each, $O(m)$ time to check $c(A,B)$

(Important) Fact. If E[X] exists, then Pr[X ≥ E[X]] > 0.

Assume $= 0$ by contradiction

Proof.

$\mu = \mathbb{E}X$

$\mu = \mathbb{E}[X(\mathbb{1}_{X<\mu} + \mathbb{1}_{X\geq\mu})]$

$= \mathbb{E}[X\mathbb{1}_{X<\mu}] + \mathbb{E}[X\mathbb{1}_{X\geq\mu}]$

$< \mathbb{E}[\mu\mathbb{1}_{X<\mu}] + \_\_\_\_$

$\leq \mu \underbrace{\Pr[X<\mu]}_{\leq 1} + \underbrace{\_\_\_\_}_{= 0}$

Method 2: Brute force 💪 via pairwise independence

Theorem. There exists a deterministic ½-approximation algorithm for Max-CUT which runs in time O(m(m+n)).

Method 3: The Method of Conditional Expectations

Idea: sequentially do the greedy choice. Sometimes it works!

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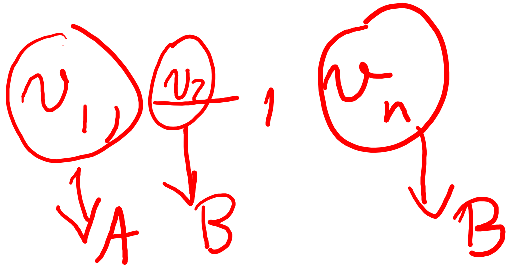
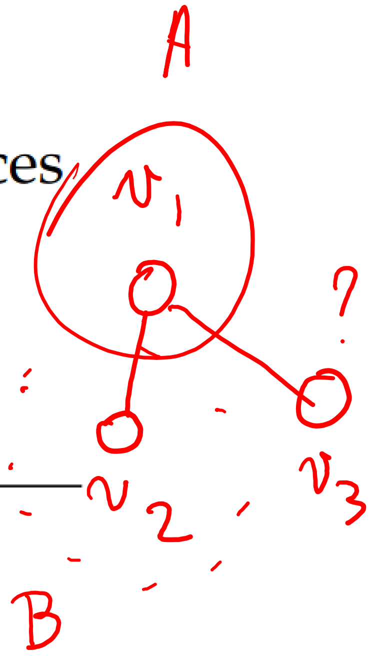
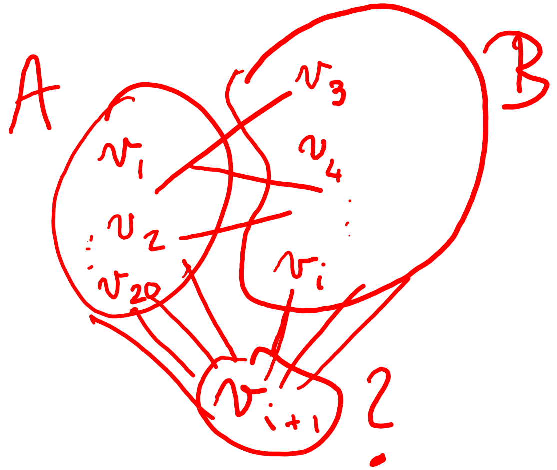




Details.



$$\frac{m}{2} \leq \mathbb{E}[c(A,B)]$$
$$\leq \mathbb{E}[c(A,B) \mid X_1]$$
$$\leq \mathbb{E}[c(A,B) \mid X_1, X_2]$$
$$\leq \dots$$
$$\leq \mathbb{E}[c(A,B) \mid X_1, \dots, X_n] = c(A,B)$$

Want $\mathbb{E}[c(A,B) \mid X_1, \dots, X_i] \leq \mathbb{E}[c(A,B) \mid X_1, \dots, X_i, X_{i+1}]$

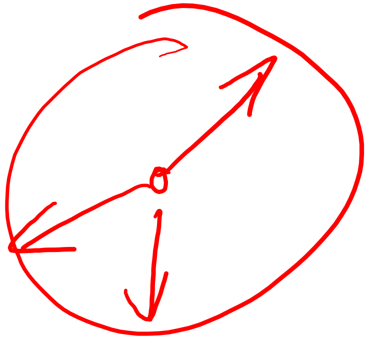
Method 3: The Method of Conditional Expectations

Theorem. There exists a deterministic ½-approximation algorithm for Max-CUT which runs in time O(mn).

Can do better!

(Aside)

Goemans - Williamson



0.878-approx.

Derandomisation: summary

- PRNG
- Brute-Force
- Pairwise (k-wise) independence
- Method of Conditional Expectations

(there is more!)

Bonus: The Probabilistic Method

"We can prove things exist without knowing how to build them."

(also can be derandomised, sometimes)