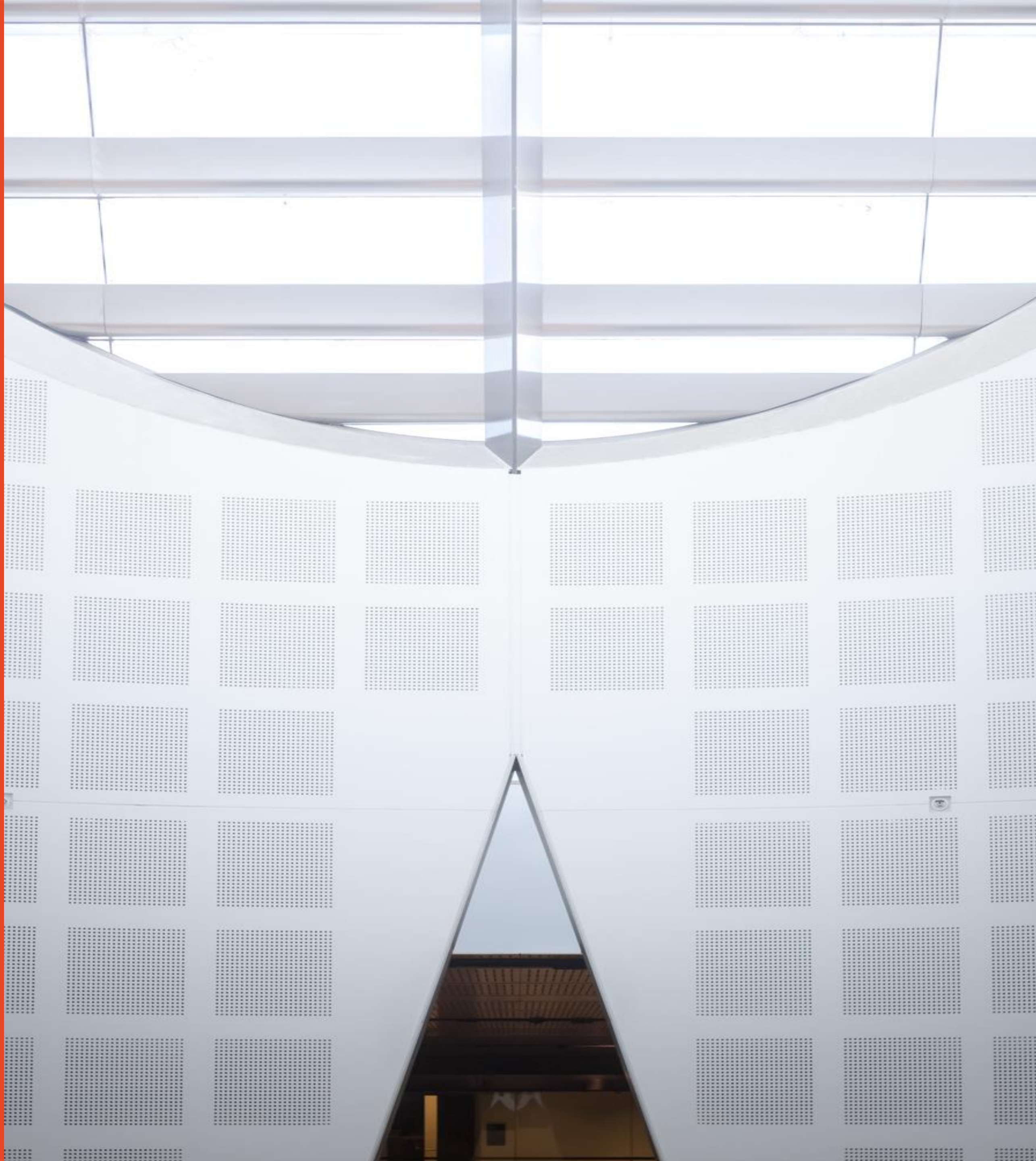
COMPx270: Randomised and Advanced Algorithms
Lecture 3: Balls in Bins

Clément Canonne

School of Computer Science

THE UNIVERSITY OF SYDNEY



A question 🎂

There are quite a few people in the classroom right now. What are the odds two of you (at least) have the same birthday?

Menti: 6529 7938

A question 🎂

Theorem. (The 🎂 paradox) If you gather 23 people in a room, then with probability at least 50% a pair will sharing their birthday.

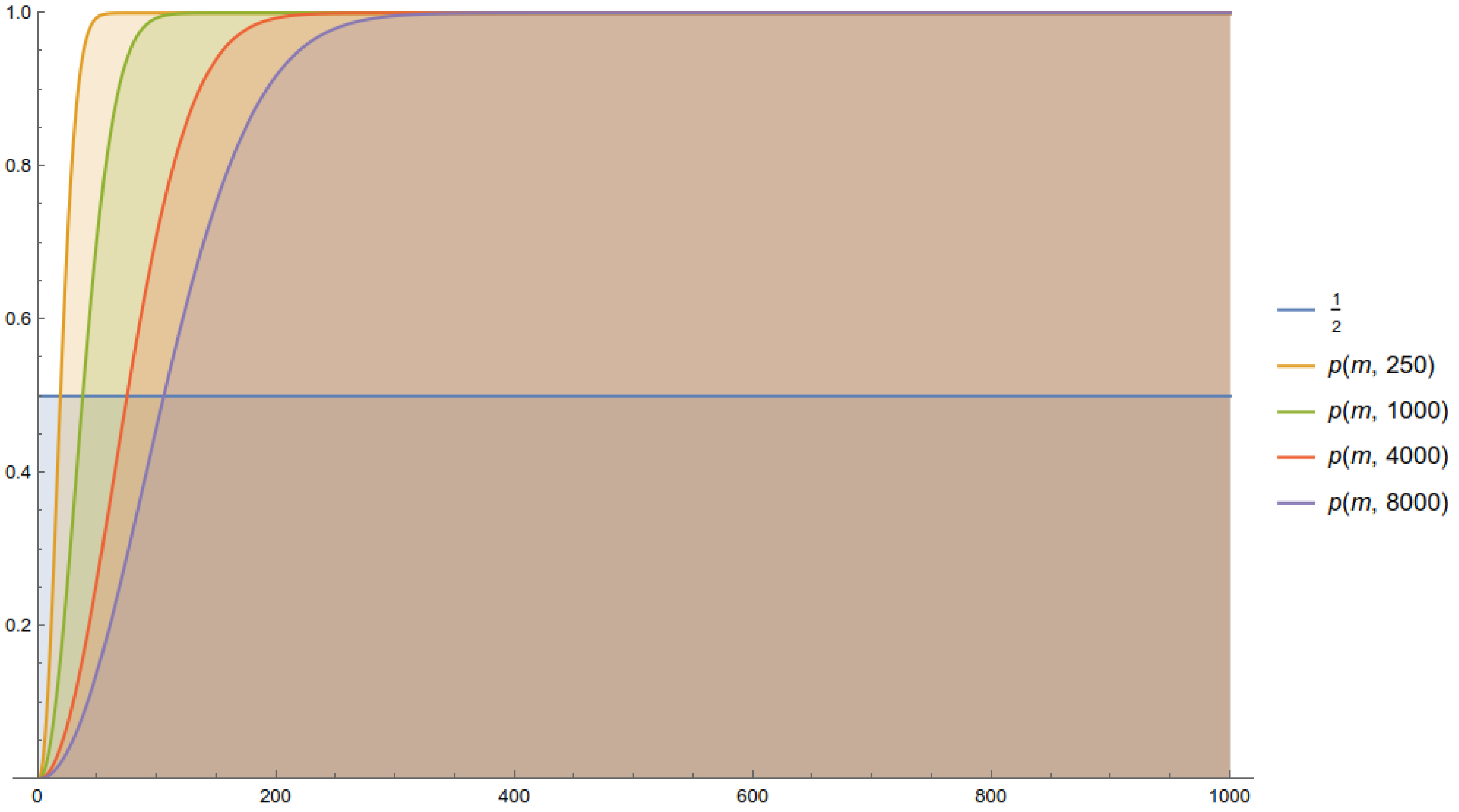
An answer? 🎂

Theorem. If you gather m people and give each a number uniform between 1 and n, then the probability p(m,n) that at least two have the same number is...

$$p_{m,n} = 1 - \frac{n!}{n^m (n-m)!} = 1 - \frac{m!}{n^m} \binom{n}{m}$$

Proof.

```
DiscretePlot[{1/2, p[m, 250], p[m, 1000], p[m, 4000], p[m, 8000]}, {m, 1, 1000}, PlotRange → {0, 1}, PlotLegends → "Expressions"]
```



Let's start simple: m=2 ⚾ ⚾

Now, for large values of "2"...

C: number of collisions when throwing m ⚾ into n 🗑️. What is

$c(m,n) = E[C]$?

Number of collisions

… and what is Var[C]?

Number of collisions

... and what is Var[C]?

Number of collisions

$$\mathrm{Var}[C] = \binom{m}{2} \frac{1}{n} \left(1 - \frac{1}{n}\right)$$

$$\mathrm{Var}[C] = \binom{m}{2}\frac{1}{n}\left(1 - \frac{1}{n}\right)$$

Number of collisions

Now, we can use Chebyshev:

$$\Pr[\,X = 0\,] \leq 1/2$$

for $m = \Omega(\sqrt{n})$.

$$\mathrm{Var}[C] = \binom{m}{2}\frac{1}{n}\left(1-\frac{1}{n}\right)$$

Number of collisions

Now, we can use Chebyshev:

$$\Pr[\, X = 0 \,] \leq 1/2$$

for $m = \Omega(\sqrt{n})$. **Is it tight?**

Number of collisions

$$\text{Var}[C] = \binom{m}{2}\frac{1}{n}\left(1 - \frac{1}{n}\right)$$

Now, we can use Chebyshev:

$$\Pr[\, X = 0 \,] \leq 1/2$$

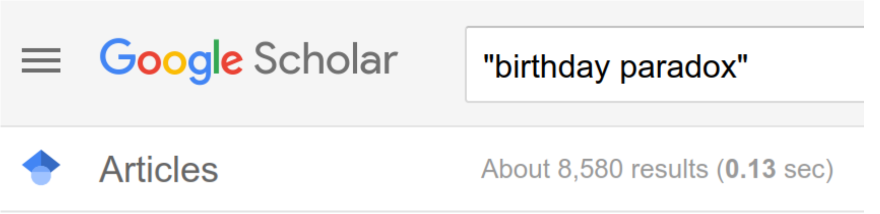
for $m = \Omega(\sqrt{n})$.

By Markov, we also have

$$\Pr[\, X \neq 0 \,] = \Pr[\, X \geq 1 \,] \leq E[X] \leq 1/2$$

for $m = O(\sqrt{n})$.

Applications?

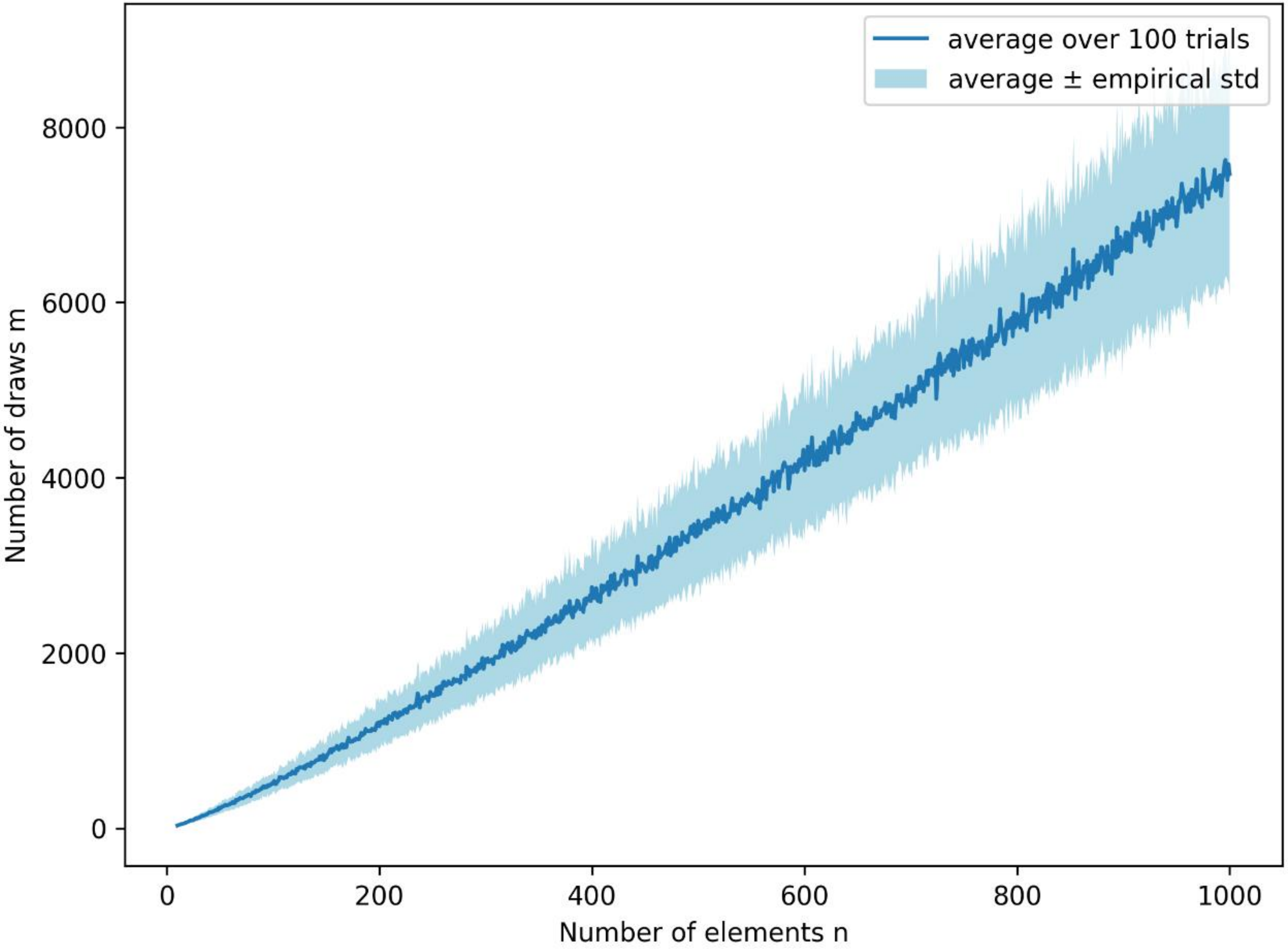


Bounding the variance: is it always that bad?

Two tricks (and even 3).

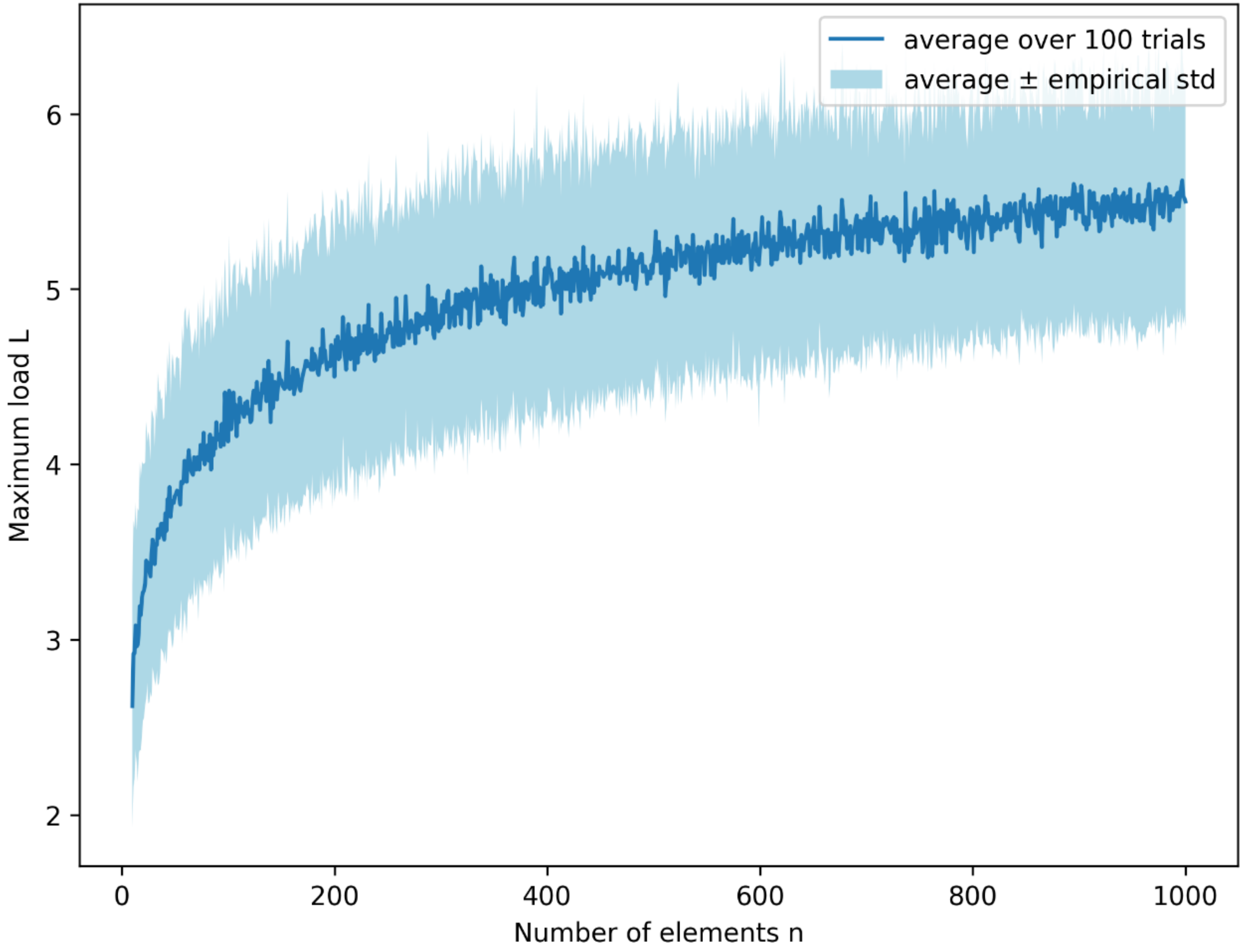
Coverage (Coupon collector)

"What is the expected number of balls $M(n)$ we need to throw before each of the n bins contains at least one ball?"


average over 100 trials
average ± empirical std
8000
6000
4000
2000
0
Number of draws m
0
200
400
600
800
1000
Number of elements n

Load balancing

"What is the expected maximum number of balls $L(n)$ any single of n bins contains when throwing n balls?"



average over 100 trials
average ± empirical std
Maximum load L
Number of elements n
2
3
4
5
6
0
200
400
600
800
1000